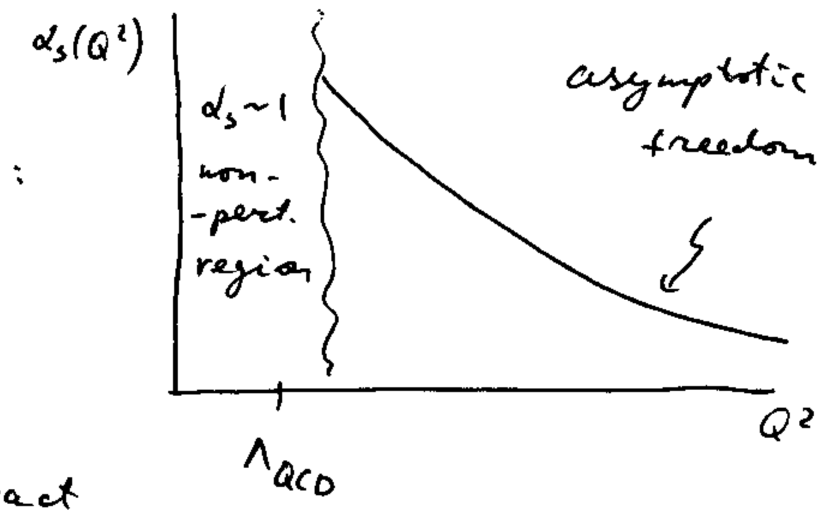
<u>Last time</u>: found the running coupling constant for QCD:

$$\alpha_s(Q^2) = \frac{\alpha_\mu}{1 + \alpha_\mu \beta_2 \ln(Q^2/\mu^2)}$$

where $\beta_2 = \frac{11 N_c - 2 N_f}{12\pi}$. Equivalently $\alpha_s(Q^2) = \frac{1}{\beta_2 \ln(Q^2/\Lambda^2)}$

with $\Lambda \simeq 200\,\text{MeV}$ a fundamental (non-perturbative) QCD scale.

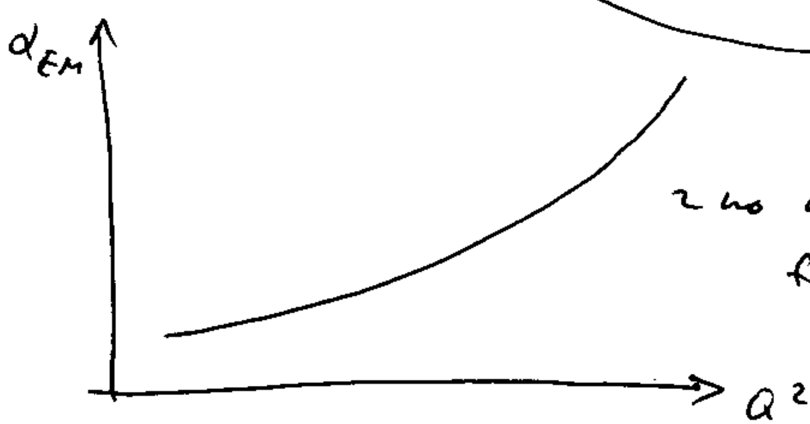


⇒ Asymptotic Freedom:

$\alpha_s \to 0$ as $Q^2 \to \infty$

⇒ at short distances quarks & gluons interact <u>weakly</u> with each other — unique property of non-abelian theories.

QED: $\beta_2 = -\frac{1}{3\pi}$ ⇒ $$\alpha_{EM}(Q^2) = \frac{\alpha_\mu}{1 - \frac{\alpha_\mu}{3\pi}\ln(Q^2/\mu^2)}$$



← no asymptotic freedom (same is true for φ^3, φ^4 theories — very common)

$\Rightarrow$ in QCD with massless quarks mesons are massless.

$\Rightarrow$ baryons have a mass. Consider proton (the lightest baryon).

proton mass: $M_p \sim$ dimensionfull quantity.

$M_p = M_p(\alpha_\mu, \mu) = \mu \, f(\alpha_\mu)$ as μ is the only dimensionfull scale.

$$\mu^2 \frac{d}{d\mu^2} M_p = 0 \Rightarrow \left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu) \frac{\partial}{\partial \alpha_\mu} \right) M_p = 0$$

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_\mu) \frac{\partial}{\partial \alpha_\mu} \right) \left[\mu f(\alpha_\mu) \right] = 0$$

$$\mu^2 \frac{\partial}{\partial \mu^2}(\mu) = \frac{1}{2} \mu \Rightarrow \left(\frac{1}{2} + \beta \frac{\partial}{\partial \alpha_\mu} \right) f(\alpha_\mu) = 0$$

$$\Rightarrow \frac{d f(\alpha_\mu)}{d\alpha_\mu} = - \frac{1}{2\beta(\alpha_\mu)} f(\alpha_\mu) \Rightarrow \frac{df}{f} = - \frac{d\alpha_\mu}{2\beta(\alpha_\mu)}$$

$$\Rightarrow \ln f(\alpha_\mu) - \ln f(\alpha_0) = -\frac{1}{2} \int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\beta(\alpha')} = -\frac{1}{2} \rho(\alpha, \alpha_0)$$

$$\Rightarrow f(\alpha_\mu) = f(\alpha_0) \, e^{-\frac{1}{2} \rho(\alpha, \alpha_0)}$$ and the

proton's mass is

$$\boxed{M_p = \mu \, f(\alpha_0) \, e^{-\frac{1}{2}\rho(\alpha_\mu, \alpha_0)}}$$

take $\beta(\alpha) = -\beta_2 \alpha^2 \Rightarrow \rho(\alpha) = \int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\beta(\alpha')} = \frac{1}{\beta_2}\left(\frac{1}{\alpha_\mu} - \frac{1}{\alpha_0}\right)$

$$\Rightarrow M_p = \mu \, f(\alpha_0) \, e^{-\frac{1}{2\beta_2}\left(\frac{1}{\alpha_\mu} - \frac{1}{\alpha_0}\right)}$$

M_p should not depend on α_0 (a cutoff) $\Rightarrow$

$\Rightarrow f(\alpha_0) \propto e^{-\frac{1}{2\beta_2}\frac{1}{\alpha_0}} \Rightarrow$ write $f(\alpha_0) = C_p e^{-\frac{1}{2\beta_2}\frac{1}{\alpha_0}}$

C_p ← constant

$\Rightarrow \boxed{M_p = C_p \cdot \mu \cdot e^{-\frac{1}{2\beta_2}\frac{1}{\alpha_\mu}}}$ ~ non-perturbative dependence on α_μ

$e^{-\frac{1}{x}}$ is a function $\neq$ to its Taylor series

$\Rightarrow$ non-perturbative!

Take $\beta(\alpha) = -\beta_2 \alpha^2 - \beta_3 \alpha^3 \Rightarrow$

pert. series ↓

$$\rho(\alpha) = -\frac{1}{\beta_2}\int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\alpha'^2\left(1 + \frac{\beta_3}{\beta_2}\alpha'\right)} = -\frac{1}{\beta_2}\int_{\alpha_0}^{\alpha_\mu} \frac{d\alpha'}{\alpha'^2}\left[1 - \frac{\beta_3}{\beta_2}\alpha' + \dots\right]$$

$$= \frac{1}{\beta_2}\left(\frac{1}{\alpha_\mu} - \frac{1}{\alpha_0}\right) + \frac{\beta_3}{\beta_2^2}\ln\frac{\alpha_\mu}{\alpha_0} + \dots$$

$$\Rightarrow M_p = \mu \, f(\alpha_0) \, e^{-\frac{1}{2}\left[\frac{1}{\beta_2}\left(\frac{1}{\alpha_\mu} - \frac{1}{\alpha_0}\right) + \frac{\beta_3}{\beta_2^2}\ln\left(\frac{\alpha_\mu}{\alpha_0}\right) + \dots\right]}$$

$\Rightarrow$ pick $f(\alpha_0) = C_p\, e^{-\frac{1}{2\beta_2\alpha_0} - \frac{\beta_3}{2\beta_2^2}\ln\alpha_0}$

$\Rightarrow$ get
$$M_p = C_p\, \mu\, \underbrace{e^{-\frac{1}{2\beta_2\alpha_\mu}}}_{\text{non-analytic ftn.}} \left(\alpha_\mu\right)^{-\frac{\beta_3}{2\beta_2^2}} \underbrace{\left(1 + o(\alpha_\mu)\right)}_{\text{analytic function}}$$

$\Rightarrow$ can not calculate M_p in perturbation theory.

Finally, $M_p = C_p\, \mu\, e^{-\frac{1}{2\beta_2\alpha_\mu}}$, remember that $\alpha_\mu = \frac{1}{\beta_2 \ln\frac{\mu^2}{\Lambda_{QCD}^2}} \Rightarrow \frac{1}{2\beta_2\alpha_\mu} = \ln\frac{\mu}{\Lambda_{QCD}}$

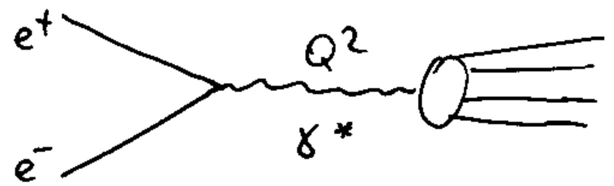
$$\Rightarrow M_p = C_p\, \mu \cdot e^{-\ln\frac{\mu}{\Lambda_{QCD}}} = C_p\, \Lambda_{QCD}$$

$\Rightarrow$ $M_p \sim \Lambda_{QCD}$ ~ a non-perturbative QCD scale where the coupling α_s is large $\Rightarrow$ can't do perturbation theory there.

The Cross Section for $e^+e^- \to$ hadrons.

$\Rightarrow$ consider e^+e^- annihilation:

$$e^+e^- \to \begin{pmatrix}\text{virtual}\\ \text{photon}\end{pmatrix} \to \text{hadrons}$$



Define the ratio $R(Q^2) = \dfrac{\sigma_{e^+e^- \to \text{hadrons}}}{\sigma_{e^+e^- \to \mu^+\mu^-}}$

$R(Q^2)$ is dimensionless $\Rightarrow R = R\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right)$

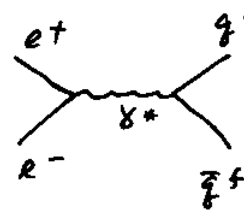
if $m_f = 0$. $\Rightarrow R = R\left(\frac{Q^2}{\mu^2}, \alpha_\mu\right) = (\text{put } \mu = Q) =$

$= R(1, \alpha(Q^2)) = R(\alpha(Q^2))$ ~ function of r.c. only

$\Rightarrow$ write a perturbative expansion for it:

$$R(\alpha(Q^2)) = R(0) + R_1\,\alpha(Q^2) + R_2\,\alpha^2(Q^2) + \ldots$$

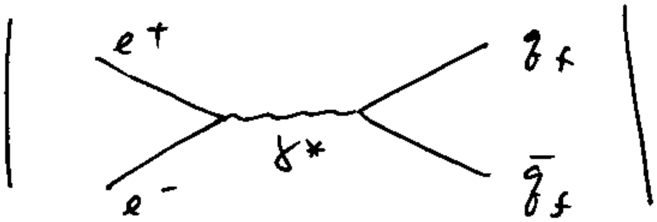
$R(0)$ is easy to get: put $\alpha(Q^2) = 0$.

$\sigma_{e^+e^- \to \text{hadrons}} \propto \Bigg|$  $\Bigg|^2 = \Bigg|$  $\Bigg|^2$

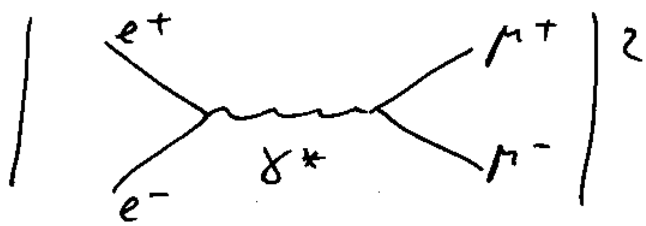
+ higher order QCD corrections

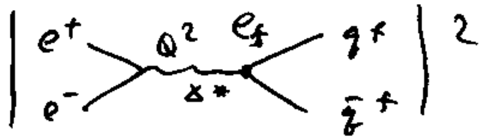
$\Rightarrow$ if $\alpha_s = 0$ $\Rightarrow$ drop higher order corrections

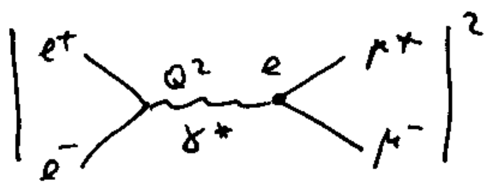
$\Rightarrow$ $\sigma_{e^+e^- \to \text{hadrons}} \simeq \sigma_{e^+e^- \to \text{quarks}} \propto$

$\propto$ 

On the other hand, with high precision

$\sigma_{e^+e^- \to \mu^+\mu^-} \propto$ 

$\Rightarrow R(0) =$ 

divided by 

$= \underset{\substack{\uparrow \\ \text{\# of} \\ \text{quark} \\ \text{colors}}}{3} \sum_f e_f^2$ — neglect q & μ masses.

Where to terminate the sum over flavors depends on Q^2: if $Q^2 < 4m_c^2$ $\Rightarrow$ $Q < 2m_c \simeq 3\,\text{GeV}$

$\Rightarrow$ need only u, d, s (3 flavors)

$$\Rightarrow R(Q < 2m_c,\ Q > 2m_s) = 3\Big(\underbrace{\left(\tfrac{2}{3}\right)^2}_{u} + \underbrace{\left(\tfrac{1}{3}\right)^2}_{d} + \underbrace{\left(\tfrac{1}{3}\right)^2}_{s}\Big) =$$

$$= 3\left(\frac{4}{9} + \frac{1}{9} + \frac{1}{9}\right) = 2$$

take $Q > 2 m_b \simeq 8.5\,GeV$ $\Rightarrow$ e.g. $Q \simeq 80\,GeV$

$$\Rightarrow R = 3\Big(\underbrace{\left(\tfrac{2}{3}\right)^2}_{u} + \underbrace{\left(\tfrac{1}{3}\right)^2}_{d} + \underbrace{\left(\tfrac{1}{3}\right)^2}_{s} + \underbrace{\left(\tfrac{2}{3}\right)^2}_{c} + \underbrace{\left(\tfrac{1}{3}\right)^2}_{b} \Big) = \frac{11}{3}$$

$\Rightarrow$ amazingly close to data (see attachement)

$\Rightarrow$ if one includes higher order corrections

get $$R(\alpha(Q^2)) = 3 \sum e_f^2 \left\{ 1 + \frac{\alpha(Q^2)}{\pi} + (1.986 - 0.115 N_f) \cdot \left(\frac{\alpha}{\pi}\right)^2 + \dots \right\}$$

$\Rightarrow$ in reality quarks become hadrons, which is a non-perturbative process ...

$\Rightarrow$ $e^+e^- \to$ hadrons gives direct evidence for quarks as fermions with 3 colors and fractional electric charges

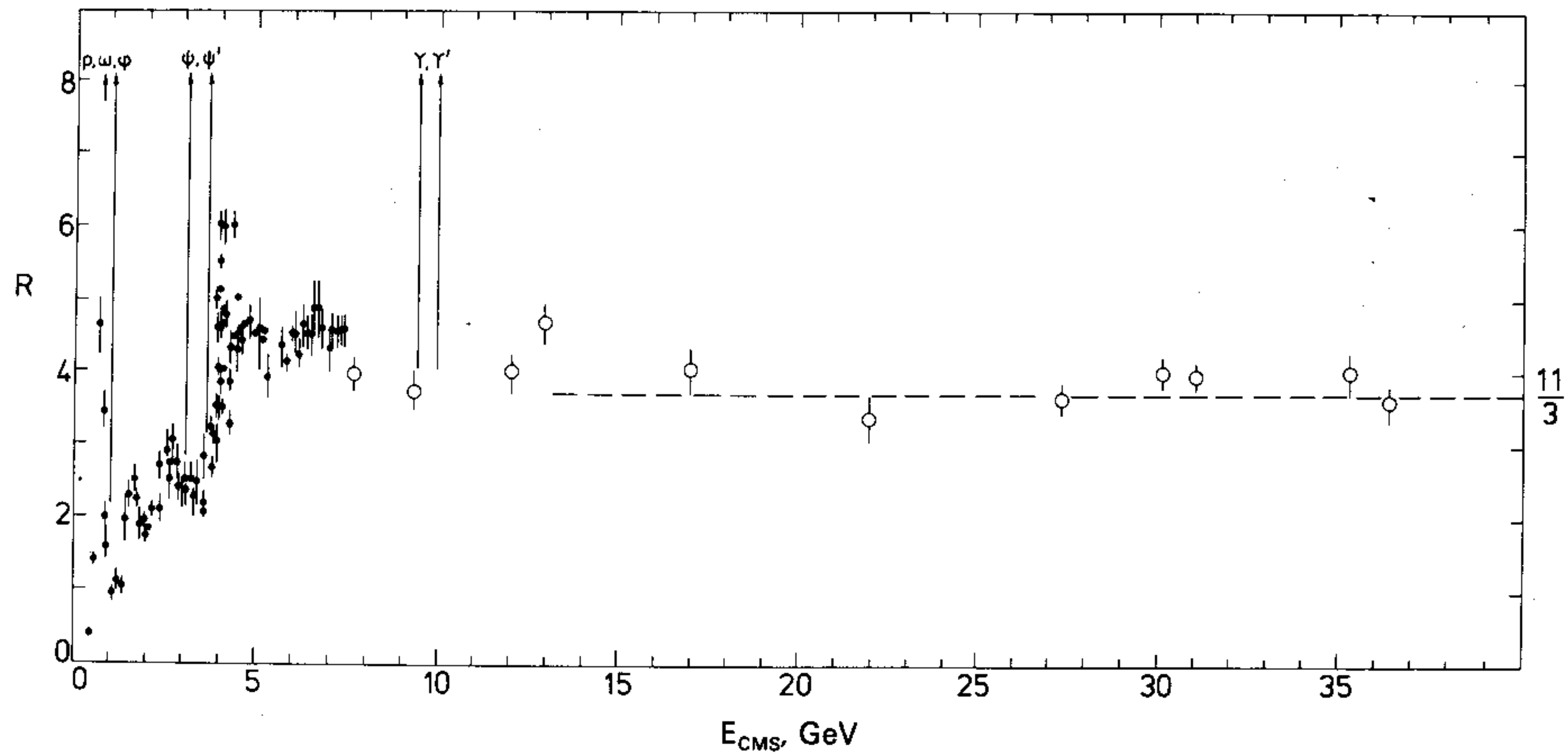


Figure 8.3 The ratio R of the cross-section for $e^+e^- \to$ hadrons, divided by that for $e^+e^- \to \mu^+\mu^-$. The fact that R is constant above 10-GeV CMS energy is a proof of the pointlike nature of hadron constituents. The predicted value of R, assuming that the primary process is formation of a quark-antiquark pair, is $\frac{11}{3}$ if pairs of u, d, s, c, b quarks are excited and they have three color degrees of freedom. The data come from many storage-ring experiments. At high energy ($>$10 GeV CMS) it is from the PETRA ring at DESY, Hamburg.

Feynman Rules in QCD

$$\boxed{\mathcal{L}_{QCD} = \bar{q}^f [i\gamma^\mu D_\mu - m_f] q^f - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}}$$

However, this Lagrangian is gauge-invariant

$$\begin{cases} A_\mu \to S A_\mu S^{-1} - \frac{i}{g} (\partial_\mu S) S^{-1} \\ q \to S q \end{cases}$$

$\Rightarrow$ need to fix the gauge!

(i) <u>Covariant (Lorenz) gauge</u> $\partial_\mu A^{a\mu} = 0$

$\Rightarrow$ to fix the gauge need to introduce the so-called ghost fields:

$$\mathcal{L}_{QCD}^{cov.\,gauge} = \bar{q}^f [i\gamma^\mu D_\mu - m_f] q^f - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^{a\mu}) (\partial_\nu A^{a\nu}) + \partial_\mu \bar{\eta} \mathcal{D}^\mu \eta$$

η^a is a scalar field ~ Faddeev-Popov ghost

η^a is an anti-commuting field (Grassmann variables) (quantized like a fermion) $\Rightarrow$ unphysical $\Rightarrow$ ghosts

$\bar{\eta}^a$ is c.c. of η