Last time: talked about proton mass: it is an essentially non-perturbative quantity.

We showed that if $\beta(\alpha) = -\beta_2 \alpha^2 - \beta_3 \alpha^3$

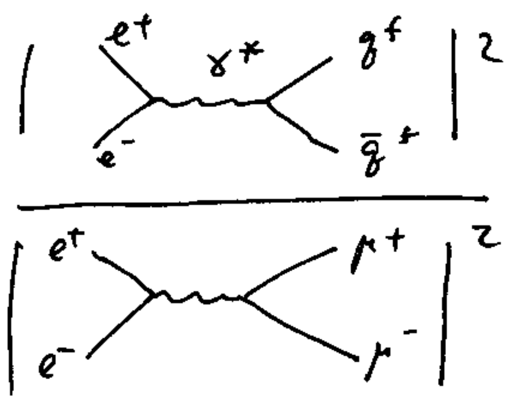
$$\Rightarrow M_p = c_p \, \mu \, \underbrace{e^{-\frac{1}{2\beta_2 \alpha_\mu}} \, (\alpha_\mu)^{-\frac{\beta_3}{2\beta_2^2}}}_{\substack{\text{non-analytic in } \alpha_\mu \\ \text{near } \alpha_\mu = 0}} \, \underbrace{\left(1 + O(\alpha_\mu)\right)}_{\text{analytic in } \alpha_\mu}$$

$\Rightarrow M_p$ is not calculable in perturbation theory!

~ an example of non-perturbative observable.

$e^+e^- \to$ hadrons:

$$R = \frac{\sigma_{e^+e^- \to \text{hadrons}}}{\sigma_{e^+e^- \to \mu^+\mu^-}} =$$



$+ O(\alpha_s)$

$$\Rightarrow R = 3 \sum_f e_f^2 .$$

$\Rightarrow$ for $2m_b < Q < 2m_t$ get $R = 11/3$ ~ agrees well with experiment!

$\Rightarrow$ quarks are "for real".

Feynman Rules in QCD

$$\boxed{\mathcal{L}_{QCD} = \bar{q}^f \left[i\gamma^\mu D_\mu - m_f \right] q^f - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}}$$

However, this Lagrangian is gauge-invariant

$$\begin{cases} A_\mu \to S A_\mu S^{-1} - \frac{i}{g} (\partial_\mu S) S^{-1} \\ q \to S q \end{cases}$$

$\Rightarrow$ need to fix the gauge!

(i) Covariant (Lorenz) gauge $\partial_\mu A^{a\mu} = 0$

$\Rightarrow$ to fix the gauge need to introduce the so-called ghost fields:

$$\boxed{\mathcal{L}_{QCD}^{\text{cov. gauge}} = \bar{q}^f \left[i\gamma^\mu D_\mu - m_f \right] q^f - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^{a\mu}) (\partial_\nu A^{a\nu}) + \partial_\mu \bar{\eta} \mathcal{D}^\mu \eta}$$

η^a is a scalar field ~ Faddeev-Popov ghost

η^a is an anti-commuting field (Grassmann variables) (quantized like a fermion) $\Rightarrow$ unphysical $\Rightarrow$ ghosts

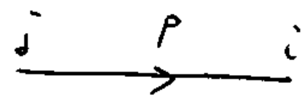
$\bar{\eta}^a$ is c.c. of η

$$\eta = \sum_{a=1}^{8} T^a \eta^a , \quad \mathcal{D}_\mu \eta = \partial_\mu \eta - ig \underbrace{[A_\mu, \eta]}_{\text{note the commutator!}}$$

$$\mathcal{D}_\mu \eta^a = \partial_\mu \eta^a + g f^{abc} A_\mu^b \eta^c$$

Feynman Rules:

Quark Propagator:



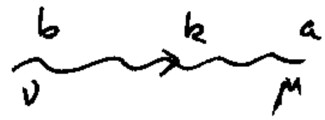
$$\frac{i}{\gamma \cdot p - m} \delta_{ij}$$

$$= \frac{i(\gamma \cdot p + m)}{p^2 - m^2 + i\varepsilon} \delta_{ij}$$

Ghost Propagator:

$$\frac{i}{k^2 + i\varepsilon} \delta_{ab}$$

Gluon Propagator:

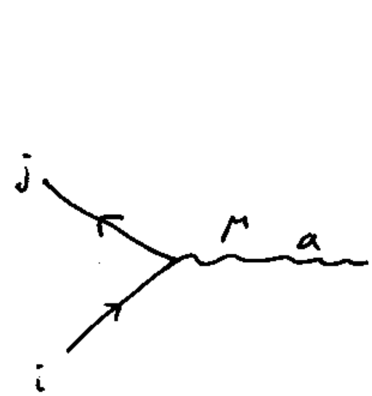


$$\frac{-i}{k^2 + i\varepsilon} \delta^{ab} \left[g_{\mu\nu} - (1-\xi) \frac{k_\mu k_\nu}{k^2} \right]$$

$\xi = 0$ Landau gauge

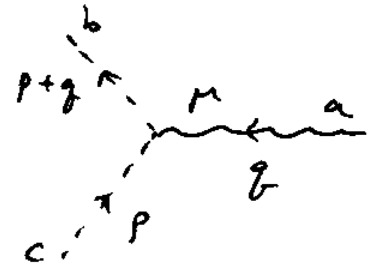
$\xi = 1$ Feynman gauge.

Quark-Gluon Vertex:



$$ig \gamma^\mu (T^a)_{ji}$$

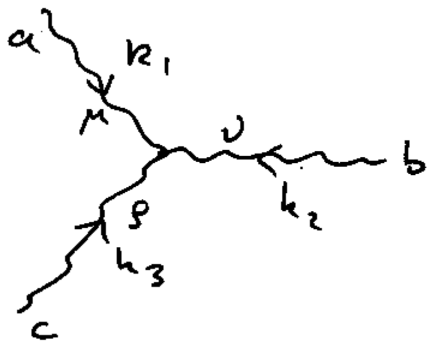
Ghost-gluon Vertex:



$$g\,(p+q)_\mu\, f^{abc}$$

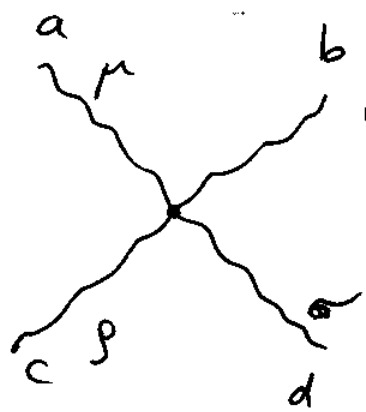
(clockwise)

3-Gluon Vertex:



$$g\, f^{abc}\,\big[(k_1-k_3)_\nu\, g_{\mu\rho} + (k_2-k_1)_\rho\, g_{\mu\nu} + (k_3-k_2)_\mu\, g_{\nu\rho} \big]$$

4-Gluon Vertex:



$$-ig^2\big[f^{abe} f^{cde} \cdot (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \big].$$

≈ same as QED for external fermions, bosons (no external ghosts), internal integrals, "−" for each fermion (or ghost) loop.

(ii) Light-cone gauge

Define light-cone variables: $A^{\pm} = \dfrac{A^0 \pm A^3}{\sqrt{2}}$

(choose a "preferred direction" $\sim x^3$)

$A^+ = 0$ gauge is called the light-cone (LC) gauge

Write the gauge condition as

$$\boxed{\eta \cdot A = 0} \quad \text{with} \quad \eta^- = 1,\ \eta^+ = 0,\ \eta^1 = \eta^2 = 0$$

$$A_\mu B^\mu = A^+ B^- + A^- B^+ - A^1 B^1 - A^2 B^2$$

(check)

$$\eta \cdot A = \underset{0}{\underset{\shortparallel}{\eta^+}} A^- + \underset{1}{\underset{\shortparallel}{\eta^-}} A^+ - \underset{0}{\underset{\shortparallel}{\eta^1}} A^1 - \underset{0}{\underset{\shortparallel}{\eta^2}} A^2 = A^+$$

=> there is no ghost in LC gauge!

Feynman rules: the same, but no ghost

=> no ghost propagator, no ghost-gluon vertex

=> gluon propagator is different:

$$\frac{-i}{k^2 + i\varepsilon}\,\delta^{ab}\left[g_{\mu\nu} - \frac{\eta_\mu k_\nu + \eta_\nu k_\mu}{\eta \cdot k}\right].$$

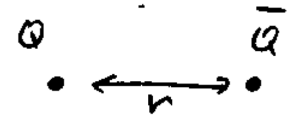
Heavy Quark Potential & Confinement.

Imagine two very heavy quarks in vacuum. Can we calculate the force one of them applies on another one?

In E&M one has Coulomb potential $V(r) \sim -\frac{\alpha_{EM}}{r}$.

Is it the same in QCD?

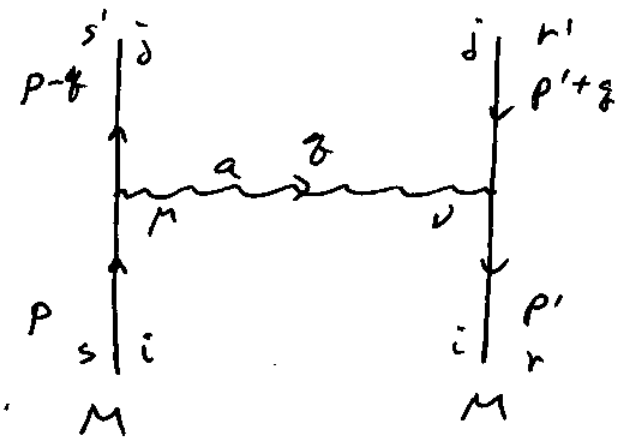
Short Distances



at small r the coupling $\alpha_s(1/r^2)$ is small

$\Rightarrow$ can do perturbation theory.

at the lowest order the potential is given by this graph:



The amplitude:

$$iM = \bar{u}_{s'}(p-q)\,\gamma^\mu\, u_s(p)\,\cdot$$

$$\cdot\, \underbrace{\bar{v}_r(p')\,\gamma^\nu\, v_{r'}(p'+q)}_{\text{need for potential}} \cdot (ig)^2 \underbrace{(T^a)_{ji}(T^a)_{ij}}_{\text{color singlet}} \frac{-i}{q^2+i\varepsilon} \underbrace{g_{\mu\nu}}_{\text{covariant gauge}}$$

$\otimes$ $\frac{d^3 q\, e^{i\vec{q}\cdot\vec{r}}}{(2\pi)^3}$ $\frac{1}{2M}$ $\cdot$ $\frac{1}{N_c}$ ~ average over colors (for potential only)

↑ quark mass

Quark mass M is very large $\Rightarrow$

$$(p-q)^2 = M^2 \Rightarrow M^2 - 2\,p\cdot q + q^2 = M^2$$

$$\Rightarrow p\cdot q \simeq M\cdot q^0 \quad \Rightarrow M^2 - 2M\cdot q^0 = M^2$$

$$(q^2 \ll p\cdot q) \Rightarrow \quad q^0 = 0 \quad \Rightarrow q^2 = -|\vec{q}\,|^2.$$

static case

$$\bar{u}_{s'}(p-q)\,\gamma^\mu u_s(p) \simeq g^{\mu 0}\cdot \bar{u}_{s'}(p-q)\,\gamma^0 u_s(p)$$

$$= g^{\mu 0}\, u^+_{s'}(p-q)\,u_s(p) = g^{\mu 0}\cdot 2M\,\delta^{ss'}$$

similarly $\bar{v}_r(p')\gamma^\nu v_{r'}(p'+q) = g^{\nu 0}\, 2M\,\delta^{rr'}$

$$iM = -g^2 \int \frac{d^3q}{(2\pi)^3}\cdot \underbrace{2M\,\delta^{ss'}\delta^{rr'}}_{\text{norm}}\;\frac{1}{\vec{q}^{\,2}}\;\underbrace{\mathrm{tr}(T^aT^a)\cdot\frac{1}{N_c}}_{\frac{N_c^2-1}{2N_c}\equiv C_F}$$

To get the potential need to turn d^3q into Fourier transform. Fixing the normalization write

choose $\vec{r} = r\,\hat{z}$ in polar coord's

$$V(r) = -g^2 C_F \int \frac{d^3q}{(2\pi)^3}\;\frac{e^{i\vec{q}\cdot\vec{r}}}{\vec{q}^{\,2}} = -g^2 C_F \int_0^\infty \frac{\cancel{q^2}\,dq}{(2\pi)^3}\,.$$

$$\cdot\int_{-1}^{1} d\cos\theta \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}\;\frac{1}{\cancel{q^2}}\,e^{iqr\cos\theta} = -g^2\frac{C_F}{(2\pi)^2}\int_0^\infty dq\,.$$

$$\cdot\frac{1}{iqr}\left(e^{iqr} - e^{-iqr}\right) = -\frac{g^2 C_F}{4\pi^2}\frac{1}{ir}\int_0^\infty \frac{dq}{q}\left(e^{iqr} - e^{-iqr}\right)$$