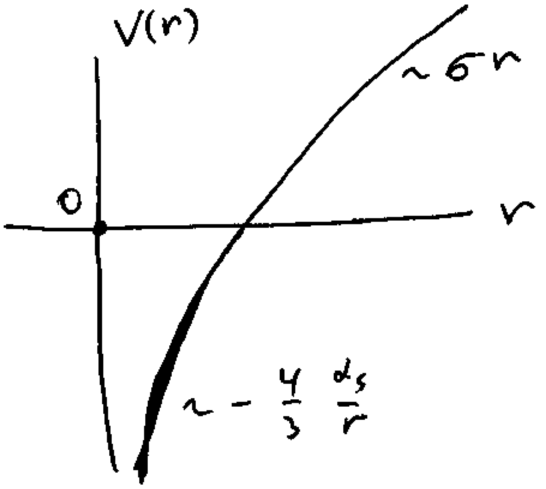
Last time: we worked out heavy quark potential:

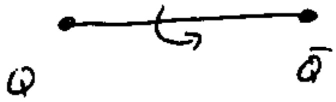
$$V(r) = \begin{cases} -\frac{4}{3}\frac{\alpha_s}{r}, & r \ll \frac{1}{\Lambda_{QCD}} \sim \text{perturbative calculation} \\ \sim \sigma r, & r \gtrsim \frac{1}{\Lambda_{QCD}} \sim \text{model + lattice calculations} \end{cases}$$

⇒



~ confining potential

⇒ talked about string model of a meson:



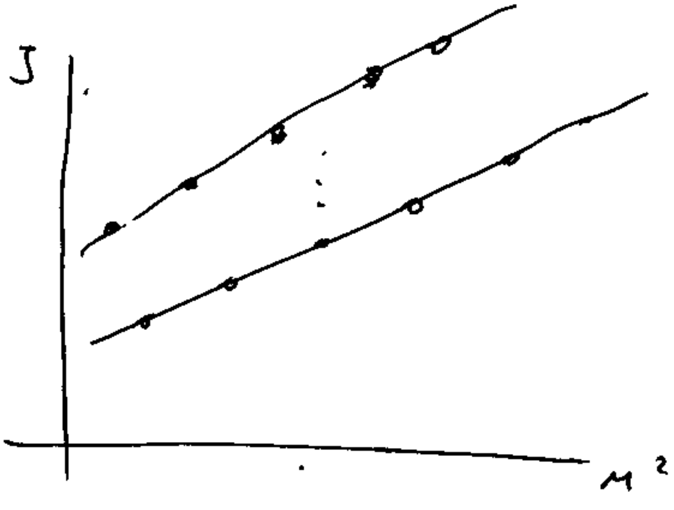
⇒ got $\boxed{J = \alpha_0 + \alpha' M_J^2}$

Regge trajectory

$\boxed{\alpha' = \frac{1}{2\pi\sigma}}$ ~ slope parameter

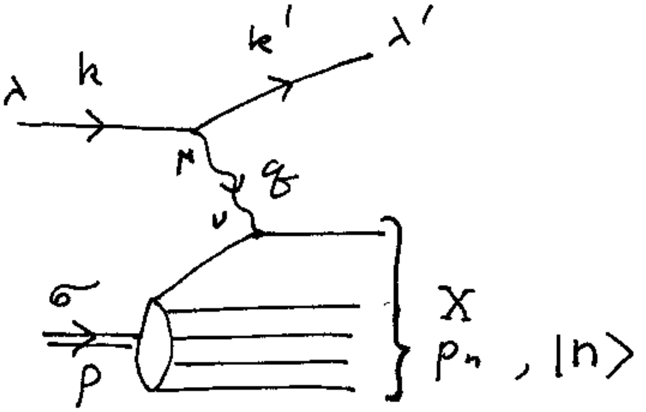
α_0 ~ intercept

⇒ supported by data:



Parton Model and DIS

Deep Inelastic Scattering. (DIS)



$$e(k) + \text{proton}(p) \to$$

$$\to e(k') + X$$

Rest frame of the proton: $p = (\overset{t}{m_p}, \overset{x}{0}, \overset{y}{0}, \overset{z}{0})$

$$k = (\varepsilon, 0, 0, k) \simeq (\varepsilon, 0, 0, \varepsilon) \quad \begin{pmatrix}\text{energy} \sim \text{many GeV} \\ \text{neglect } m_e\end{pmatrix}$$

$$k' = (\varepsilon', \varepsilon' \sin\Theta, 0, \varepsilon' \cos\Theta)$$

Define:

$$\to Q^2 \equiv -q^2 = -(k-k')^2 \simeq 2k\cdot k' = 2\varepsilon\varepsilon'(1-\cos\theta) =$$

$$= 4\varepsilon\varepsilon' \sin^2\frac{\theta}{2}$$

$$\nu \equiv \frac{p\cdot q}{m} = \varepsilon - \varepsilon' \quad \leftarrow \text{only } \nearrow \text{in } p\text{'s rest frame}$$

$$\to x \equiv \frac{Q^2}{2p\cdot q} = \frac{Q^2}{2m\nu} \quad \text{Bjorken } x \text{ variable}$$

$$x = \frac{Q^2}{2p\cdot q} = \frac{-q^2}{(p+q)^2 - p^2 - q^2}$$

$$\hat{s} \equiv (p+q)^2 \Rightarrow x = \frac{Q^2}{Q^2 + \hat{s}} = \frac{Q^2}{\hat{s} - m_p^2 + Q^2} \simeq \frac{Q^2}{\hat{s} + Q^2} \quad \text{if } Q^2 \gg m_p^2$$

Q^2 and x are important / independent !

Interaction amplitude:

$$T_{\sigma,\lambda,\lambda'}(n) = +ie\,\bar{u}_{\lambda'}(k')\,\gamma_\mu\,u_\lambda(k)\,\frac{-ig^{\mu\nu}}{q^2}\cdot$$

$$\cdot\, ie\,\langle n|j_\nu(0)|p,\sigma\rangle$$

where $j_\nu(x) = \sum_f e_f\,\bar{q}_f(x)\,\gamma_\nu\,q_f(x)$

with $e_f = +2/3,\ -1/3,\ \ldots$ (quark flavors electric charges)

j_ν is EM current

Let's calculate the cross-section:

$$d\sigma = \underset{\substack{\nearrow\\ \text{spin averaging}}}{\frac{1}{4}}\sum_{\sigma,\lambda,\lambda'}\sum_n |T_{\sigma,\lambda,\lambda'}(n)|^2\,(2\pi)^4\delta^4(q+p-p_n)\cdot$$

$$\cdot\,\frac{d^3k'}{2\mathcal{E}\;2\mathcal{E}'(2\pi)^3} = \frac{e^4}{Q^4}\cdot$$

$$\overbrace{\frac{1}{2}\sum_{\lambda,\lambda'}\left[\bar{u}_{\lambda'}(k')\,\gamma_\mu\,u_\lambda(k)\right]^*\left[\bar{u}_{\lambda'}(k')\,\gamma_\nu\,u_\lambda(k)\right]}^{l_{\mu\nu}}\cdot$$

$$\underbrace{\cdot\,\frac{1}{2}\sum_{\sigma,n}\langle n|j^\mu(0)|p,\sigma\rangle^*\,\langle n|j^\nu(0)|p,\sigma\rangle\cdot\,(2\pi)^4\,\delta(q+p-p_n)}_{=\ \frac{m(2\pi)^3}{4\pi^2E_p}W^{\mu\nu}}\;\frac{d^3k'}{2\mathcal{E}\;2\mathcal{E}'}$$

Therefore

$$\boxed{\frac{d\sigma}{d^3k'} = \frac{e^4}{Q^4\ 4\,\varepsilon\cdot\varepsilon'}\ \frac{1}{4\pi^2}\ l_{\mu\nu}\ W^{\mu\nu}}$$

$(E_p = m)$

$$l_{\mu\nu} = \frac{1}{2}\sum_{\lambda,\lambda'} \bar{u}^*_{\lambda'\alpha}(k')\left(\gamma_\mu^*\right)_{\alpha\beta} u^*_{\lambda\beta}(k)\ \bar{u}_{\lambda'\alpha'}(k')\cdot$$

see next page

$$\cdot(\gamma_\nu)_{\alpha'\beta'}\ u_{\lambda\beta'}(k) = \frac{1}{2}\sum_{\lambda,\lambda'} \bar{u}_\lambda(k)\ \gamma_\mu\ u_{\lambda'}(k')\cdot$$

$$\cdot\bar{u}_{\lambda'}(k')\ \gamma_\nu\ u_\lambda(k) = \frac{1}{2}\,\mathrm{Tr}\left[\gamma_\mu\ \gamma\cdot k'\ \gamma_\nu\ \gamma\cdot k\right]$$

as $\sum_{\lambda'} u_{\lambda'}(k')\,\bar{u}_{\lambda'}(k') = \gamma\cdot k' + m_e \simeq \gamma\cdot k'$

Using $\mathrm{Tr}\left[\gamma_\alpha\gamma_\beta\gamma_\mu\gamma_\nu\right] = 4\left[g_{\alpha\beta}\,g_{\mu\nu} + g_{\alpha\nu}\,g_{\beta\mu} -\right.$

$\left. - g_{\alpha\mu}\,g_{\beta\nu}\right]$

We get

$$\boxed{l_{\mu\nu} = 2\left(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}\ k\cdot k'\right)}$$

$$W_{\mu\nu} = \frac{4\pi^2 E_p}{m}\ \frac{1}{2}\sum_{\sigma,n} \langle n|\,j^\mu(0)\,|p,\sigma\rangle^* \langle n|\,j^\nu(0)\,|p,\sigma\rangle$$

$$\cdot(2\pi)^4\ \delta^4\left(q+p-p_n\right) = \frac{4\pi^2 E_p}{m}\ \frac{1}{2}\sum_{\sigma,n}\int d^4x\ e^{iq\cdot x}\ .$$

$$\langle p,\sigma|\,\underset{\substack{\| \\ e^{ip\cdot x}\,j^\mu(0)\,e^{-ip\cdot x}}}{j^\mu(x)}\,|n\rangle\ \langle n|\,j^\nu(0)\,|p,\sigma\rangle$$

(Heisenberg picture)

$$\left[\bar{u}_{\lambda'}(k')\gamma^\mu u_\lambda(k)\right]^* \underset{\substack{\uparrow \\ \text{as this is a scalar}}}{=} \left[u^+_{\lambda'}\gamma^0\gamma^\mu u_\lambda\right]^+ =$$

$$= u^+_\lambda \underbrace{(\gamma^0)^2}_{\mathbb{1}} \gamma^{+\mu}\gamma^{+0} u_{\lambda'} = \bar{u}_\lambda(k)\gamma^0\gamma^{+\mu}\gamma^{+0}u_{\lambda'}$$

now, $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \Rightarrow \gamma^{+0} = \gamma^0 \Rightarrow \gamma^0\gamma^{+\mu}\gamma^{+0} = \gamma^0\gamma^{+\mu}\gamma^0$

Let's find $\gamma^0\gamma^{+\mu}\gamma^0$:

$\mu = 0 \Rightarrow \gamma^0\gamma^{+0}\gamma^0 = \gamma^0$

$\mu = i \Rightarrow \gamma^0\gamma^{+i}\gamma^0 = -\gamma^0\gamma^i\gamma^0 = \gamma^i(\gamma^0)^2 =$

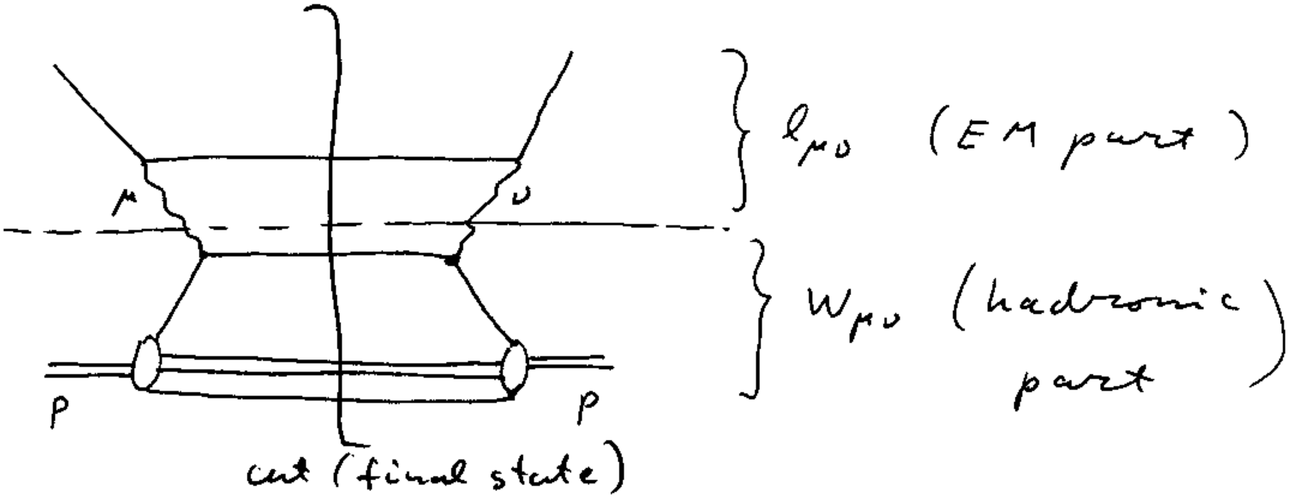
$= \gamma^i$

$\Rightarrow \boxed{\gamma^0\gamma^{+\mu}\gamma^0 = \gamma^\mu}$

$\Rightarrow$ get $\bar{u}_\lambda(k)\gamma^\mu u_{\lambda'}(k')$ as desired.

$$\boxed{W_{\mu\nu} = \frac{4\pi^2 E_p}{m}\int d^4x\, e^{iq\cdot x}\langle p|\, j_\mu(x)\, j_\nu(0)\,|p\rangle}$$

(averaged over σ)



$$W_{\mu\nu}(p,q) = A(x,Q^2)\,p_\mu p_\nu + B(x,Q^2)\,q_\mu q_\nu + C(x,Q^2)\,g_{\mu\nu} +$$

$$D(x,Q^2)\,(p_\mu q_\nu + p_\nu q_\mu) + E(x,Q^2)\,(p_\mu q_\nu - p_\nu q_\mu) +$$

$$+ F(x,Q^2)\,\varepsilon_{\mu\nu\rho\sigma}\,p^\rho q^\sigma$$

$F = 0$ in $\gamma^* p$, $\gamma^* A$ (F comes from γ_5's, appears in ν DIS).

(1) $q_\mu W^{\mu\nu} = 0$ (current conservation)

$q_\mu \to \partial_\mu \Rightarrow \partial_\mu j^\mu(x) = 0$

$$A\,p_\nu\,(p\cdot q) + B\,q_\nu\,q^2 + C\,q_\nu + D\,(p\cdot q\; q_\nu + q^2 p_\nu)$$

$$+ E\,(p\cdot q\; q_\nu - q^2 p_\nu) = 0$$

(2) $$q_\nu W^{\mu\nu} = 0 = A\,p_\mu\,(p\cdot q) + B\,q^2 q_\mu \overset{+Cq_\mu}{\vee} + D(p\cdot q\; q_\mu +$$

$$+ q^2 p_\mu) + E\,(p_\mu q^2 - p\cdot q\; q_\mu) = 0$$