Last time: We showed that the most general $W_{\mu\nu}$ (hadronic tensor in DIS) is written as:

$$\boxed{W_{\mu\nu} = -W_1(x,Q^2)\left[g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right] + \frac{W_2(x,Q^2)}{m_p^2}\left[p_\mu - \frac{p\cdot q}{q^2}q_\mu\right]\cdot\left[p_\nu - \frac{p\cdot q}{q^2}q_\nu\right]}$$

(we imposed current conservation $q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$)

$W_1, W_2 \sim$ structure functions

DIS cross section: $$\boxed{\frac{d\sigma}{d^3k'} = \frac{4\alpha_{EM}^2}{Q^4}\left[2W_1\sin^2\left(\frac{\theta}{2}\right) + W_2\cos^2\left(\frac{\theta}{2}\right)\right]}$$

More conventional: $F_1 = m_p W_1$, $F_2 = \nu W_2$

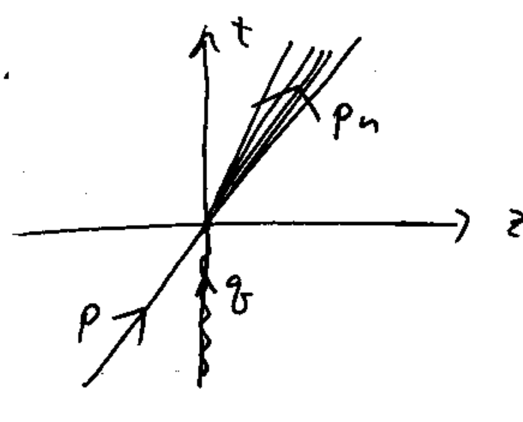
$$\boxed{\frac{d\sigma}{d^3k'} = \frac{4\alpha_{EM}^2}{Q^4}\left[\frac{2}{m_p}F_1\sin^2\left(\frac{\theta}{2}\right) + \frac{2m_p x}{Q^2}F_2\cos^2\left(\frac{\theta}{2}\right)\right]}$$

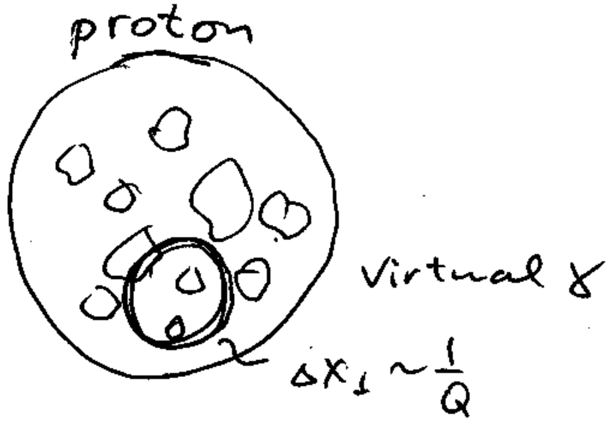
The Parton Model (cont'd)

IMF ~ infinite momentum frame: $p_\mu \simeq (p + \frac{m_p^2}{2p}, 0, 0, p)$

$q_\mu \simeq (q^0, \vec{q}, 0)$

$Q^2 \simeq \vec{q}^{\,2}$ => virtual photon resolves transverse distance $\boxed{\Delta x_\perp \simeq \frac{1}{Q}}$

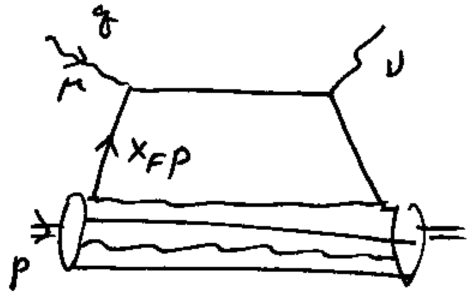




<u>Defined</u> "forward amplitude"

$$\boxed{T_{\mu\nu} = \frac{\mathcal{E}_p}{2\pi m_p} \int d^4x \, e^{iq\cdot x} \, \frac{1}{2} \sum_\sigma \langle p, \sigma | \, T \, j_\mu(x) j_\nu(0) \, | p, \sigma \rangle}$$

$T_{\mu\nu} =$



(no cut, forward amplitude)

$$\boxed{\tau_{DIS} \simeq \frac{1}{q^0} \simeq \frac{x_{Bj} P}{2Q^2} \ll \tau_\Lambda \simeq \frac{P}{m_p} \frac{1}{\Lambda_{QCD}}}$$

interaction is very quick compared to cross-talk between quarks & gluons in the proton!

$\Rightarrow$ q's & g's are "frozen" in time in the IMF.

<u>Optical theorem</u>:

$$\boxed{W_{\mu\nu} = 2 \, \mathrm{Im} \, (i \, T_{\mu\nu})}$$

if x is small ($\lesssim 1$) and Q is large

$\Rightarrow$ $\tau_{DIS} \ll \tau_\Lambda$ interaction is "instantaneous".

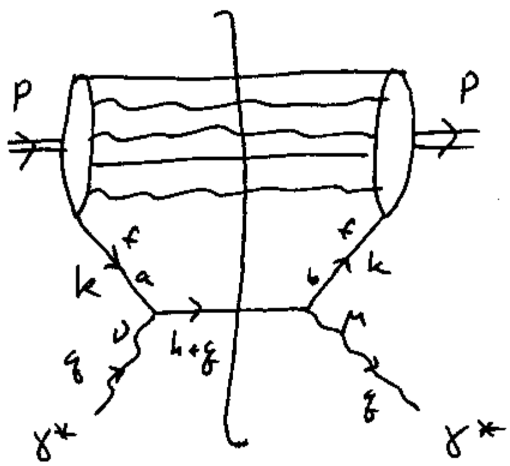
Define light cone variables:

for vector V^μ one has $V^+ = \frac{1}{\sqrt{2}}(V^0 + V^3)$

$\underline{V} = (V^1, V^2)$ $\qquad V^- = \frac{1}{\sqrt{2}}(V^0 - V^3)$

(2d transverse vector)

$V_1 \cdot V_2 = V_{1\mu} V_2^\mu = V_{1+} V_{2-} + V_{1-} V_{2+} - \underline{V}_1 \cdot \underline{V}_2$

$W_{\mu\nu} =$ 

$\Big\}\ A^f_{ab}(p,k)$

e.g.

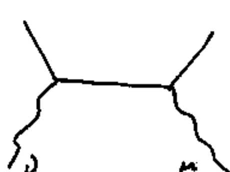
$p_+ = \frac{p}{\sqrt{2}}$

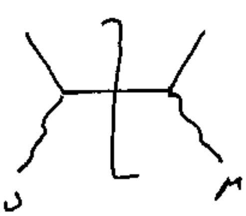
$p_- = \frac{m^2}{2\sqrt{2}\,p}$

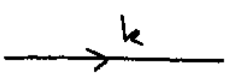
$p_+ \gg p_-$

as $W_{\mu\nu} = 2\,\mathrm{Im}\,(i\,T_{\mu\nu})$

cut (Im part of a propagator)

$T_{\mu\nu} =$ 

$W_{\mu\nu} =$ 



$\dfrac{i}{k^2 - m^2 + i\varepsilon}$ $\Rightarrow$ 

$2\pi\,\delta^{(+)}(k^2 - m^2)$

as $2\,\mathrm{Im}\,\dfrac{-1}{k^2 - m^2 + i\varepsilon} = +2\pi\,\delta^{(+)}(k^2 - m^2)$

We write

$$W_{\mu\nu} = \frac{4\pi^2 E_p}{\cancel{(2\pi)^3} m} \sum_f e_f^2 \int d^4k \; A_{ab}^f(p,k) \left[\gamma_\mu \gamma\cdot(k+q)\gamma_\nu\right]_{ba}$$

$\cdot \cancel{(2\pi)}\, \delta((k+q)^2)$ where A_{ab}^f is the rest of the diagram (see p.46).

Start calculating assuming that

$$Q^2 \gg k^2, \; \underline{k}\cdot\underline{q} \quad , \quad k_+ \gg k_- \; (IMF)$$

$$(k+q)^2 = k^2 + 2k_+ q_- + 2k_- q_+ - \underline{k}\cdot\underline{q} - Q^2$$

$q_3 = 0 \Rightarrow q_+ = q_- \Rightarrow$ as $k_+ \gg k_- \Rightarrow$ drop $2k_- q_+$

dropping $k^2, \underline{k}\cdot\underline{q} \ll Q^2$ get

$$(k+q)^2 \simeq 2k_+ q_- - Q^2$$

$$\Rightarrow \delta((k+q)^2) \simeq \delta(2k_+ q_- - Q^2) = \delta\left(\frac{k_+}{p_+} 2p_+ q_- - Q^2\right)$$

as $p\cdot q \simeq \cancel{2} p_+ q_- \Rightarrow$ and $x_{Bj} = \frac{Q^2}{2p\cdot q}$

$$\Rightarrow \delta((k+q)^2) \simeq \frac{x_{Bj}}{Q^2} \delta\left(x_{Bj} - \frac{k_+}{p_+}\right)$$

$$\Rightarrow \boxed{x_{Bj} = \frac{k_+}{p_+}}$$ Feynman x = Bjorken x

physical meaning: light cone momentum fraction of struck quark!

$$\gamma_0 (k+q) = \gamma_+ (k_- + q_-) + \gamma_- (k_+ + q_+) - \underline{\gamma} \cdot (\underline{k} + \underline{q})$$

after d^4k : $\gamma_+ \to p_+ \quad \gamma_- \to p_- \quad \underline{\gamma} \to \underline{p} = 0$

$\Rightarrow$ as $p_+ \gg p_-$ keep γ_+ only, $q_- \simeq \frac{Q^2}{x \cdot 2p_+}$, $(k_- \ll q_-)$

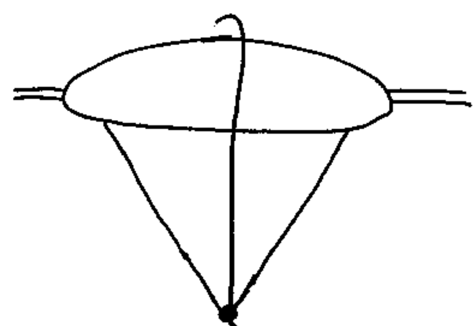
$$W_{\mu\nu} = \frac{E_p}{2 m p_+} \sum_f e_f^2 \int d^4k \; A^f_{ab}(p,k) \left[\gamma_\mu \gamma_+ \gamma_\nu \right]_{ba} \cdot \delta\left(x - \frac{k_+}{p_+} \right)$$

symmetrize, as $W_{\mu\nu}$ is symmetric

Concentrate on $W_{ij} \sim \frac{1}{2} \left[\gamma_i \gamma_+ \gamma_j + \gamma_j \gamma_+ \gamma_i \right] =$

$= -\frac{1}{2} \gamma_+ \{ \gamma_i, \gamma_j \} = -g_{ij} \gamma_+$ (we used $W_{ij} = W_{ji}$)

DIS now looks like



$\gamma_+ \, \delta\left(x - \frac{k_+}{p_+}\right)$

(Mueller vertex)

We have $W_{ij} \propto g_{ij}$ from diagram calculations.

On the other hand, since $\underline{p} = 0$

$$W_{ij} = -W_1 \left(g_{ij} - \frac{q_i q_j}{q^2} \right) + \frac{W_2}{m_p^2} q_i q_j \left(\frac{p \cdot q}{q^2} \right)^2 =$$

$$= -W_1 g_{ij} + \frac{q_i q_j}{q^2} \left[W_1 + \frac{W_2}{m_p^2} \frac{(p \cdot q)^2}{q^2} \right] \propto g_{ij}$$

$$\Rightarrow \quad W_1 + \frac{W_2}{m^2}\frac{(p\cdot q)^2}{q^2} = 0$$

as $\nu = \frac{p\cdot q}{m}$ and $x = \frac{Q^2}{2p\cdot q} = -\frac{q^2}{2p\cdot q}$

we write $\boxed{\nu W_2 = 2 m x W_1}$ Callan-Gross Relation '69

follows from spin-$\frac{1}{2}$ nature of quarks!

(would be different for particles with different spin); equivalently: $\boxed{F_2(x,Q^2) = 2x F_1(x,Q^2)}$

Exercise: show that Callan-Gross relation leads to $\frac{d\sigma}{d^3k'} \sim \left[1 + \left(1 - \frac{\nu}{\varepsilon}\right)^2\right] W_1$

CG relation leads to

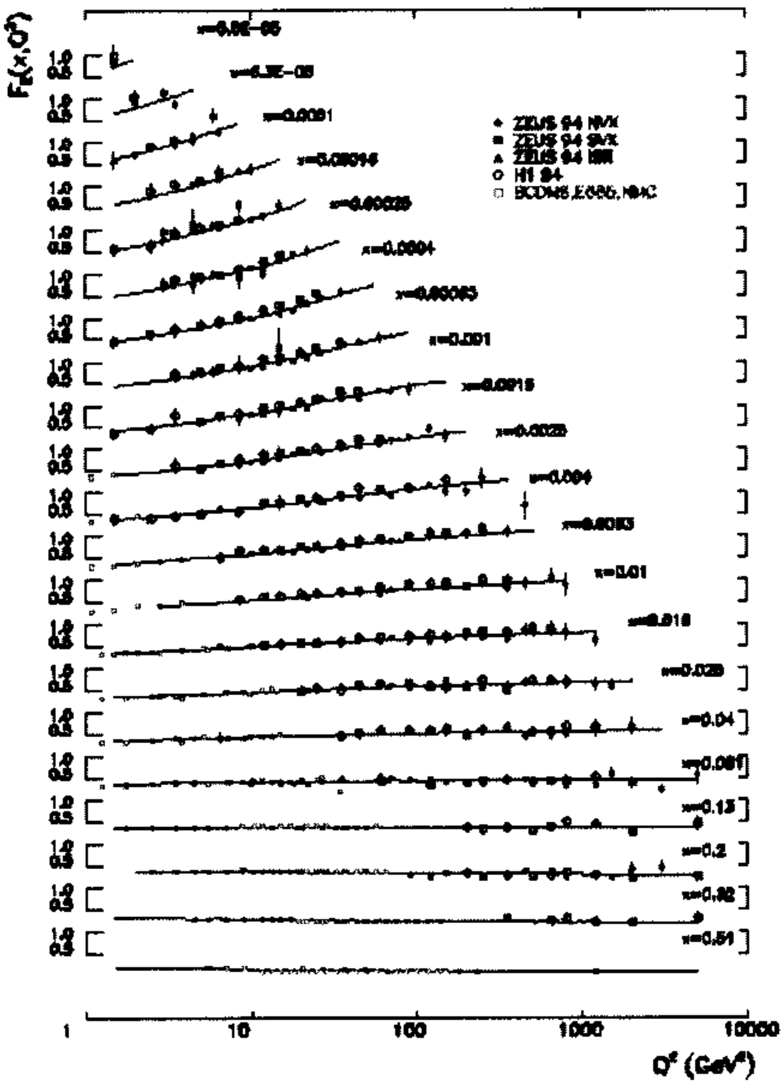
$$\nu W_2 = 2mx W_1 = \cancel{2}\cancel{m}x \cdot \frac{E_p}{\cancel{2}\cancel{m}p_+}\sum_f e_f^2 \int d^4k\, A^f_{ab}(p,k)\cdot$$

$$\cdot(\gamma_+)_{ba}\,\delta\left(x - \frac{k_+}{p_+}\right) \Rightarrow$$ <u>defining quark distribution</u>:

$$\boxed{P^f(x) \equiv \frac{E_p}{p_+}\int d^4k\, A^f_{ab}(p,k)\,(\gamma_+)_{ba}\,\delta\left(x - \frac{k_+}{p_+}\right)}$$

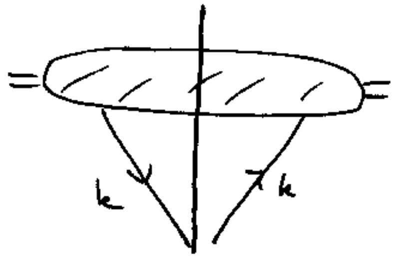
we get $\boxed{\underset{(x,Q^2)}{\nu W_2} = \sum_f e_f^2\, x P^f(x)}$ no Q^2-dependence only x-dependent <u>Bjorken scaling</u> (see attached)

Bjorken scaling was first measured at SLAC in 1968: it killed string models and brought back field theories.



Structure function $F_2(x, Q^2)$ plotted as a function of Q^2 for various values of x. Note that at large-x (lower curves) it is Q^2-independent; this is Bjorken scaling!

at low-x (upper curves) Bjorken scaling is violated.

$P^f(x) = \dfrac{E_p}{p+}$  $\Rightarrow$ often $P^f(x)$ is denoted $q(x)$.

$\gamma_+ \delta(x - \frac{k_+}{p_+})$

$\Rightarrow P^f(x, Q^2)$ counts # of quarks with light cone momentum x and transverse momentum $k_T \lesssim Q$.

parton distribution function $(P^f \sim a^+ a)$

$\Rightarrow$ for a free quark $A^f_{ab}(p,k)(\gamma_+)_{ba} = \delta^4(p-k) \dfrac{1}{2E_p}$.

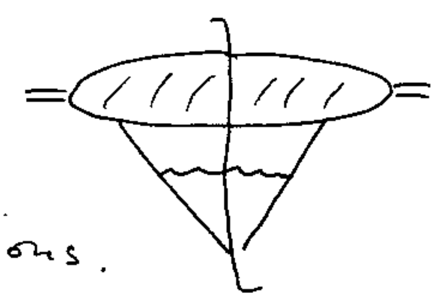
$\underbrace{\dfrac{\bar{u}_b(p)(\gamma_+)_{ba} u_a(p)}{2E_p}}_{=2p_+}$ ...

$\dfrac{\bar{u}_b(p)(\gamma+)_{ba}u_a(p)}{}$ $= \dfrac{p_+}{E_p} \delta^4(p-k)$ $\overset{\text{plug in.}}{\Rightarrow}$ $P^f(x) = \delta(x-1)$

one quark at $x = 1$

Peskin, ch. 17.5
Sterman ch 14

QCD Improved Parton Model: DGLAP equation

How about corrections like  ?

These are important corrections.

However, let us first discard the negligible diagrams like