Last time: solved Klein-Gordon equation

$$[\partial_\mu \partial^\mu + m^2]\varphi = 0 \Rightarrow \varphi = \int \frac{d^3k}{(2\pi)^3 2E_k}\left[a_{\vec{k}} e^{-ik\cdot x} + a^*_{\vec{k}} e^{ik\cdot x}\right]$$

with $E_k = \sqrt{\vec{k}^2 + m^2}$. We found that $\pm E_{\vec{k}}$ are both solutions $\Rightarrow$ negative particle states. $\Rightarrow$ need to quantize the system as a field!

Canonical Quantization: define canonical momentum

$$\pi = \frac{\delta \mathcal{L}}{\delta(\partial_0 \varphi)}$$

Commutation relations:

$$\begin{cases} [\varphi(\vec{x},t), \pi(\vec{x}',t)] = i\,\delta(\vec{x}-\vec{x}') \\ [\varphi(\vec{x},t), \varphi(\vec{x}',t)] = [\pi(\vec{x},t), \pi(\vec{x}',t)] = 0 \end{cases}$$

Hamiltonian $\hat{H} = \int d^3x\,[\pi\dot{\varphi} - \mathcal{L}]$, $\quad i\dfrac{\partial \hat{O}}{\partial t} = [\hat{O}, \hat{H}]$

time evolution of operators

free scalar field: $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2$

$$\Rightarrow \hat{H} = \int d^3x\left[\frac{1}{2}(\partial_0\varphi)^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2\right] = \int\frac{d^3k}{(2\pi)^3 2E_k} E_k\left[\hat{N}(\vec{k}) + \infty\right]$$

$\hat{N}(\vec{k}) = \hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}}$ ~ particle # operator.

Free Dirac Field (cont'd): $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In Weyl representation one has

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3$$

where σ^i's are Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Def. $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ One can show that these $\sigma_{\mu\nu}$ matrices are generators of Lorentz algebra.

Def. $\psi_\alpha(x)$, $\alpha = 1, 2, 3, 4$ is called a spinor if it transforms as

$$\psi(x) \to \psi'(x') = e^{-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}\psi(x)$$

under Lorentz transformation. $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$

(E.g. $\omega^{0i} = \xi^i \sim$ boosts, $\omega^{12} = \theta^3 \sim$ rotations, etc.)

Def. $\bar{\psi}_\alpha = \psi^+_\beta (\gamma_0)_{\beta\alpha}$, $\quad \psi^+ = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$

One can show that $\bar{\psi}(x) \to \bar{\psi}'(x') = \bar{\psi}(x)\, e^{\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}$.

Therefore $\bar{\psi}\psi \equiv \sum_{\alpha=1}^{4} \bar{\psi}_\alpha \psi_\alpha$ is a Lorentz-invariant!

One can show that $\bar{\psi}\gamma^\mu\psi$ is a 4-vector

using $\boxed{e^{\frac{i}{4}\omega^{\alpha\beta}\sigma_{\alpha\beta}}\,\gamma^\mu\,e^{-\frac{i}{4}\omega^{\rho\sigma}\sigma_{\rho\sigma}} = \Lambda^\mu{}_\nu\gamma^\nu}$: ← Lorentz transformation matrix

$$\bar{\psi}_{(x)}\gamma^\mu\psi_{(x)} \to \bar{\psi}'(x')\gamma^\mu\psi'(x') = \bar{\psi}(x)\,e^{\frac{i}{4}\omega^{\alpha\beta}\sigma_{\alpha\beta}}\cdot\gamma^\mu\cdot e^{-\frac{i}{4}\omega^{\rho\sigma}\sigma_{\rho\sigma}}\psi(x) = \Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\psi.$$

$\Rightarrow$ free Dirac field lagrangian is

$$\boxed{\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi}$$

~ it is a Lorentz-scalar, m is the mass.

We have a lagrangian $\mathcal{L} = \mathcal{L}(\psi, \bar{\psi}, \partial_\mu\psi)$

$\Rightarrow$ EOM are $\dfrac{\delta\mathcal{L}}{\delta\bar{\psi}_\alpha} - \partial_\mu\underbrace{\dfrac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{\psi}_\alpha)}}_{=0} = 0$

$\Rightarrow$ get $\boxed{\left[i\gamma^\mu\partial_\mu - m\right]\psi = 0}$ Dirac equation.

~ can make it $\psi, \bar{\psi}$ ~ symmetric ~ add/subtract a divergence of $\frac{1}{2}\bar{\psi}\,i\gamma^\mu\psi$. $\Rightarrow \mathcal{L} = \frac{1}{2}\bar{\psi}\,i\gamma^\mu\partial_\mu\psi - \frac{1}{2}(\partial_\mu\bar{\psi})\,i\gamma^\mu\psi - m\bar{\psi}\psi$

Let's analyze Dirac equation:

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0 \quad \Rightarrow \text{apply } i\gamma^\nu \partial_\nu \quad \Rightarrow$$

$$\Big[-\underbrace{\gamma^\nu \partial_\nu \gamma^\mu \partial_\mu}_{=} - m\, i\gamma^\nu \partial_\nu\Big]\psi = 0$$

$$\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu = \frac{1}{2}\{\gamma^\nu, \gamma^\mu\}\partial_\mu \partial_\nu = g^{\mu\nu}\partial_\mu \partial_\nu = \partial_\mu \partial^\mu$$

$$\Rightarrow \Big[-\partial_\mu \partial^\mu - m \underbrace{i\gamma^\mu \partial_\mu}_{= m \text{ by Dirac equation}}\Big]\psi = 0$$

$$\Rightarrow \quad \left[\partial_\mu \partial^\mu + m^2\right]\psi = 0$$

$\Rightarrow$ if the field satisfies Dirac equation, it also satisfies Klein-Gordon equation! $\Rightarrow$ also has < 0 energy "particles"

$\Rightarrow$ Write the solution as

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3\, 2E_k}\left[e^{-ik\cdot x}\,\psi^{(+)}(\vec{k}) + e^{ik\cdot x}\,\psi^{(-)}(\vec{k})\right]$$

& plug back into the original Dirac equation:

$\partial_\mu \to -ik_\mu$ in the 1st term, $+ik_\mu$ in the second

$\Rightarrow$ get
$$(\gamma\cdot k - m)\,\psi^{(+)}(\vec{k}) = 0$$
$$(\gamma\cdot k + m)\,\psi^{(-)}(\vec{k}) = 0$$

$\Rightarrow$ write $\psi^{(+)} = \begin{pmatrix} \psi^{(+)}_u \\ \psi^{(+)}_\ell \end{pmatrix}$

$$\Rightarrow \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \Rightarrow$$

$$\gamma\cdot k = \gamma^0 k_0 + \gamma^i k_i = \gamma_0 k_0 - \vec{\gamma}\cdot\vec{k} = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}$$

$$- \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{k} \\ -\vec{\sigma}\cdot\vec{k} & 0 \end{pmatrix} = \begin{pmatrix} \varepsilon & -\vec{\sigma}\cdot\vec{k} \\ \vec{\sigma}\cdot\vec{k} & -\varepsilon \end{pmatrix}$$

$$\Rightarrow (\gamma\cdot k - m)\,\psi^{(+)} = \begin{pmatrix} \varepsilon - m & -\vec{\sigma}\cdot\vec{k} \\ \vec{\sigma}\cdot\vec{k} & -\varepsilon - m \end{pmatrix} \begin{pmatrix} \psi_u^{(+)} \\ \psi_\ell^{(+)} \end{pmatrix} = 0$$

$$\begin{cases} (\varepsilon - m)\,\psi_u^{(+)} - \vec{\sigma}\cdot\vec{k}\,\psi_\ell^{(+)} = 0 \\ \vec{\sigma}\cdot\vec{k}\,\psi_u^{(+)} - (\varepsilon + m)\,\psi_\ell^{(+)} = 0 \end{cases} \Rightarrow \psi_\ell^{(+)} = \frac{\vec{\sigma}\cdot\vec{k}}{\varepsilon + m}\,\psi_u^{(+)}$$

~ solves the whole thing (why?)

$$\Rightarrow \psi^{(+)} = \begin{pmatrix} \psi_u^{(+)} \\ \frac{\vec{\sigma}\cdot\vec{k}}{\varepsilon+m}\,\psi_u^{(+)} \end{pmatrix}$$

$\Rightarrow$ reduced a 4-component unknown spinor to 2 unknown components

Similarly
$$\psi^{(-)} = \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{\varepsilon+m}\,\psi_\ell^{(-)} \\ \psi_\ell^{(-)} \end{pmatrix}.$$

Choose a basis: $\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and

<u>define</u>

$$u_r(\vec{k}) = \sqrt{\varepsilon+m}\begin{pmatrix} \chi_r \\ \frac{\vec{\sigma}\cdot\vec{k}}{\varepsilon+m}\,\chi_r \end{pmatrix}; \quad v_r(\vec{k}) = \sqrt{\varepsilon+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{\varepsilon+m}\,\chi_r \\ \chi_r \end{pmatrix}$$

then we write

$$\boxed{\psi(x) = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3\, 2E_k} \left\{ \hat{b}_{\vec{k},r}\, u_r(\vec{k})\, e^{-ik\cdot x} + \hat{d}^{+}_{\vec{k},r}\, v_r(\vec{k})\, e^{ik\cdot x} \right\}}$$

Canonical quantization: $\pi = \dfrac{\delta \mathcal{L}}{\delta(\partial_0 \psi)} = \bar{\psi}\, i\gamma^0 =$

$= \psi^+ \gamma_0 \gamma^0 \cdot i = i\psi^+$ as $\gamma_0\gamma^0 = \mathbb{1}$.

promote $\hat{b}$ & $\hat{d}$ to operators (note that $(i\gamma^\mu \partial_\mu - m)\psi = 0$ still holds!)

$\Rightarrow H = \int d^3x \left[\pi \dot{\psi} - \mathcal{L} \right] = \int d^3x \cdot \left[i\psi^+ \dot{\psi} - \mathcal{L} \right]$

$= \int d^3x \left[i\bar{\psi}\gamma^0 \partial_0 \psi - \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \right] =$

$= \int d^3x \left[\cancel{i\bar{\psi}\gamma^0 \partial_0 \psi} \cancel{- \bar{\psi}\, i\gamma^0 \partial_0 \psi} + i\bar{\psi}\gamma^i \partial_i \psi + \right.$

$\left. + \bar{\psi}\psi m \right] = \int d^3x\ \bar{\psi} \underbrace{\left[i\gamma^i \partial_i + m \right] \psi}_{i\gamma^0 \partial_0 \psi \text{ (Dirac eqn.)}}$

$= \int d^3x\ i\psi^+ \partial_0 \psi \Rightarrow \boxed{H = \int d^3x\ i\psi^+ \partial_0 \psi}$

H is not ≥ 0 at the classical level $\sim$ problem!

$\Rightarrow$ this is cured by quantization!

Plug in the solution of Dirac equation into the Hamiltonian:

$$\psi^+ = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \left\{ \hat{b}^+_{\vec{k},r}\, u^+_r(\vec{k})\, e^{ik\cdot x} + \hat{d}_{\vec{k},r}\, v^+_r(\vec{k})\, e^{-ik\cdot x} \right\}$$

$$\Rightarrow H = \int d^3x\, i\psi^+ \partial_0 \psi = \int d^3x \sum_{r,r'=1}^{2} \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \frac{d^3k'}{(2\pi)^3 2\mathcal{E}_{k'}}\, i\,\cdot$$

$$\cdot \left[\hat{b}^+_{\vec{k}',r'}\, u^+_{r'}(\vec{k}')\, e^{ik'\cdot x} + \hat{d}_{\vec{k}',r'}\, v^+_{r'}(\vec{k}')\, e^{-ik'\cdot x} \right] \cdot \left[\hat{b}_{\vec{k},r}\, u_r(\vec{k}) \cdot \right.$$

$$\left. \cdot(-i\mathcal{E}_k)\, e^{-ik\cdot x} + \hat{d}^+_{\vec{k},r}\, v_r(\vec{k})\, (i\mathcal{E}_k)\, e^{ik\cdot x} \right]$$

① $\hat{b}^+\hat{b}$ -term: $\int d^3x\, e^{ik'\cdot x - ik\cdot x} = (2\pi)^3\, \delta^3(\vec{k}-\vec{k}')$

$$\Rightarrow \text{get} \sum_{r,r'=1}^{2} \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \frac{d^3k'}{(2\pi)^3 2\mathcal{E}_k}\, \mathcal{E}_k\, (2\pi)^3 \delta(\vec{k}-\vec{k}')\, \hat{b}^+_{\vec{k}',r'}\, \hat{b}_{\vec{k},r}$$

$$u^+_{r'}(\vec{k}')\, u_r(\vec{k}) = \sum_{r,r'=1}^{2} \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k}\, \frac{1}{2}\, \hat{b}^+_{\vec{k},r'}\, \hat{b}_{\vec{k},r}\, u^+_{r'}(\vec{k})\, u_r(\vec{k})$$

$$\Rightarrow u_r(\vec{k}) = \sqrt{\mathcal{E}_k+m} \begin{pmatrix} \chi_r \\ \frac{\vec{\sigma}\cdot\vec{k}}{\mathcal{E}_k+m}\chi_r \end{pmatrix} \Rightarrow u^+_{r'} u_r = (\mathcal{E}_k+m)\left[\chi^T_{r'}\cdot\chi_r + \right.$$

$$\left. + \chi^T_{r'} \frac{\overset{=\vec{\sigma}}{\vec{\sigma}^+}\cdot\vec{k}}{\mathcal{E}_k+m} \frac{\vec{\sigma}\cdot\vec{k}}{\mathcal{E}_k+m}\chi_r \right] = (\mathcal{E}_k+m)\left[\delta_{rr'} + \frac{\vec{k}^2}{(\mathcal{E}_k+m)^2}\delta_{rr'} \right]$$

$$= \delta_{rr'} \frac{1}{\mathcal{E}_k+m}\left[(\mathcal{E}_k+m)^2 + \overbrace{\mathcal{E}_k^2 - m^2}^{\vec{k}^2} \right] = \delta_{rr'} \frac{1}{\mathcal{E}_k+m}\left(2\mathcal{E}_k^2 + 2\mathcal{E}_k m \right) =$$

$$= 2\mathcal{E}_k \delta_{rr'}$$

$$\Rightarrow \hat{b}^\dagger \hat{b}\text{-term} = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \, \varepsilon_k \, \hat{b}^+_{\vec{k},r} \, \hat{b}_{\vec{k},r}.$$

(2) $\hat{b}^+ \hat{d}^+$ term: $\int d^3x \, e^{ik'\cdot x + ik\cdot x} = e^{2i\varepsilon_k \cdot t} (2\pi)^3 \delta(\vec{k}+\vec{k}')$

$\Rightarrow$ get $\propto u^+_{r'}(-\vec{k}) \, v_r(\vec{k})$

$$v_r(\vec{k}) = \sqrt{\varepsilon_k + m} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{\varepsilon_k + m} \chi_r \\ \chi_r \end{pmatrix} \Rightarrow u^+_{r'}(-\vec{k}) \, v_r(\vec{k}) =$$

$$= (\varepsilon_k + m) \begin{pmatrix} \chi^T_{r'} & \chi^T_{r'} \frac{\vec{\sigma}^+ \cdot (-\vec{k})}{\varepsilon_k + m} \end{pmatrix} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{\varepsilon_k + m} \chi_r \\ \chi_r \end{pmatrix} =$$

$$= \chi^T_{r'} (\vec{\sigma}\cdot\vec{k}) \chi_r - \chi^T_{r'} (\vec{\sigma}\cdot\vec{k}) \chi_r = 0$$

In the end get

$$\boxed{H = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \, \varepsilon_k \left[\hat{b}^+_{\vec{k},r} \hat{b}_{\vec{k},r} - \hat{d}_{\vec{k},r} \hat{d}^+_{\vec{k},r} \right]}$$

Still not positive definite? Really, if we define some commutation relation for $\hat{d}, \hat{d}^+$ $\Rightarrow$ would get $\hat{b}^+ \hat{b} - \hat{d}^+ \hat{d}$ ~ not good!

Define anti-commutation relations:

$$\boxed{\begin{aligned} \{\hat{b}_{\vec{k},r}, \hat{b}^{+}_{\vec{k}',r'}\} &= \{\hat{d}_{\vec{k},r}, \hat{d}_{\vec{k}',r'}\} = (2\pi)^3\, 2\mathcal{E}_k\, \delta_{rr'}\, \delta^3(\vec{k}-\vec{k}') \\ \{\hat{b}_{\vec{k},r}, \hat{b}_{\vec{k}',r'}\} &= \{\hat{b}^{+}_{\vec{k},r}, \hat{b}^{+}_{\vec{k}',r'}\} = 0 \\ \{\hat{d}_{\vec{k},r}, \hat{d}_{\vec{k}',r'}\} &= \{\hat{d}^{+}_{\vec{k},r}, \hat{d}^{+}_{\vec{k}',r'}\} = 0 \end{aligned}}$$

where $\{\hat{A},\hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ = anti-commutator

$\Rightarrow$ dropping ∞ number get

$$\boxed{H = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3\, 2\mathcal{E}_k}\, \mathcal{E}_k \left[\hat{b}^{+}_{\vec{k},r}\, \hat{b}_{\vec{k},r} + \hat{d}^{+}_{\vec{k},r}\, d_{\vec{k},r} \right]}$$

Now it's positive-definite!

For the fields get $\{\psi_\alpha(\vec{x},t), \underbrace{\pi_\beta(\vec{x}',t)}_{= i\psi_\beta^{+}}\} = i\delta_{\alpha\beta}\,\delta(\vec{x}-\vec{y})$

$$\{\psi_\alpha, \psi_\beta\} = \{\psi^{+}_\alpha, \psi^{+}_\beta\} = 0$$

$\Rightarrow$ all operators anti-commute.

Time evolution:
$$\left.\begin{aligned} +i\frac{\partial}{\partial t}\psi(x) &= [\psi, H] \\ i\frac{\partial}{\partial t}\bar{\psi}(x) &= [\bar{\psi}, H] \end{aligned}\right\}$$ Still uses commutators (can show)