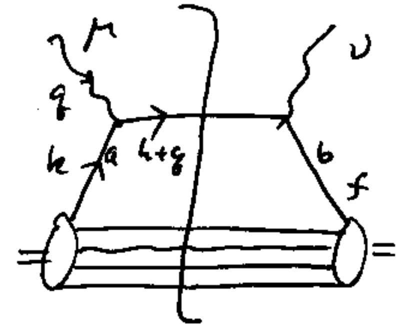
Last time: calculated

$W_{\mu\nu} =$ 


Demanded that $k+q$-line is on mass shell & got

$$x_{Bj} = \frac{Q^2}{2p\cdot q} = x_F = \frac{k^+}{p^+}$$

Bjorken-x = Feynman-x

Evaluating $W_{\mu\nu}$ ($W_{ij}$, $i,j=1,2$) we got

$$\nu W_2(x) = 2m\, x\, W_1(x)$$

or, equivalently, $\boxed{F_2(x) = 2x\, F_1(x)}$

Callan-Gross relations, prove that partons coupling to $\gamma^*$ are quarks (spin-$\frac{1}{2}$ fermions).

We defined quark distribution:

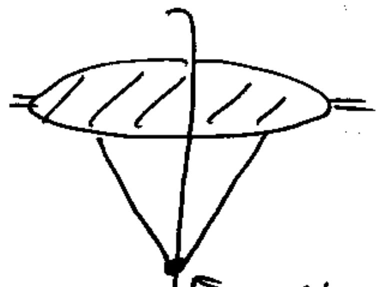
$$q^f(x) = \frac{\mathcal{E}_p}{p^+} \int d^4k\, A^f_{ab}(p,k)\, (\gamma^+)_{ba}\, \delta\left(x - \frac{k^+}{p^+}\right)$$ $\Rightarrow$ showed that

$$\Rightarrow \boxed{\nu W_2 = \sum_f e_f^2\, x\, q^f(x)} \text{ or } \boxed{F_2 = \sum_f e_f^2\, x\, q^f(x)}$$

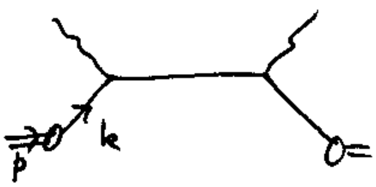
$\Rightarrow$ $F_2$ depends on $x$ only! Bjorken scaling

$\Rightarrow$ note that $F_2$ counts the # of quarks

$q^f(x, Q^2)$ = # of quarks with light cone momentum fraction $x$ and with transverse momentum $k_T \leq Q$.

$$q^f(x) = \frac{\mathcal{E}_p}{p^+}$$ 


DIS on a single quark:




$$A^f_{ab}(p,k)\,(\gamma^+)_{ba} = \delta^4(p-k)\,\frac{1}{2\mathcal{E}_p}\,\underbrace{\bar{u}(p)\gamma^+ u(p)}_{\simeq 2p^+}$$

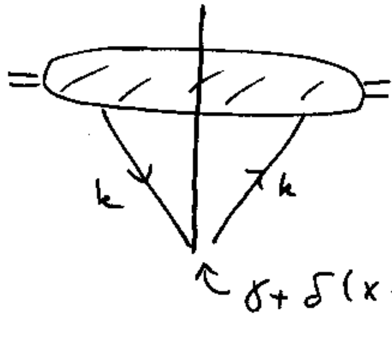
$$= \frac{p^+}{\mathcal{E}_p}\,\delta^4(p-k)$$

$$\Rightarrow q^f(x) = \frac{\mathcal{E}_p}{p^+}\int d^4k\; A^f_{ab}(p,k)\,(\gamma^+)_{ba}\;\delta\!\left(x - \frac{k^+}{p^+}\right) =$$

$$= \cancel{\frac{\mathcal{E}_p}{p^+}}\,\cancel{\int d^4k}\;\cancel{\frac{p^+}{\mathcal{E}_p}}\;\cancel{\delta^4(p-k)}\;\delta\!\left(x - \frac{k^+}{p^+}\right) = \delta(x-1)$$

$$\Rightarrow \boxed{q^f(x, Q^2) = \delta(x-1)}$$ have 1 quark at $x = 1$.

$$F_2 = \sum_f e_f^2\, x q^f(x) \;\Rightarrow\; F_2 = e_f^2\,\delta(x-1)$$

$p^f(x) = \dfrac{E_p}{p_+}$ 


$\Rightarrow$ often $p^f(x)$ is denoted $q(x)$.

$\Rightarrow p^f(x, Q^2)$ counts # of quarks with light cone momentum $x$ and transverse momentum $k_T \lesssim Q$.

parton distribution function $\quad (p^f \sim a^+ a)$

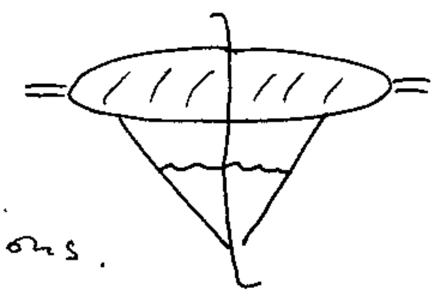
$\Rightarrow$ for a free quark $A^f_{ab}(p,k)(\gamma_+)_{ba} = \delta^4(p-k)\dfrac{1}{2E_p}$.

$\underbrace{\bar{u}_b(p)(\gamma_+)_{ba} u_a(p)}_{=2p_+} \Big/ 2p_+ \;\cdot\; = \dfrac{p_+}{E_p}\,\delta^4(p-k)$ $\overset{\text{plug in.}}{\Rightarrow}$ $p^f(x) = \delta(x-1)$

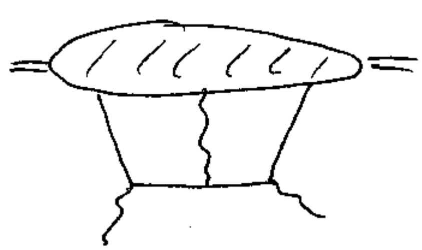
one quark at $x=1$

Peskin, ch. 17.5
Sterman ch 14

## QCD Improved Parton Model; DGLAP equation

How about corrections like  ?

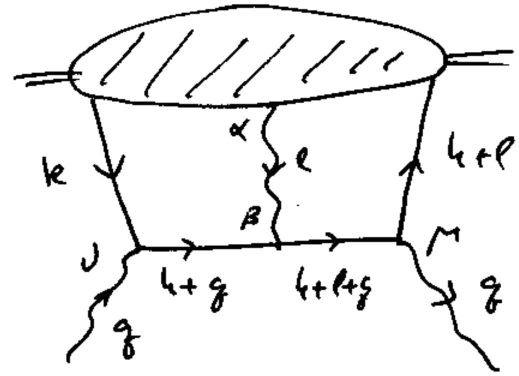
These are important corrections.

However, let us first discard the negligible diagrams like 

Work in Light Cone (LC)

gauge: $\eta \cdot A = A^+ = 0$

$(\eta^+ = 0, \eta^- = 1, \underline{\eta} = 0)$




$Q^2 \sim$ very large:

$$\Gamma_{\mu\nu\beta} = \gamma_\mu \frac{\gamma\cdot(k+l+q)}{(k+l+q)^2 + i\varepsilon} \gamma_\beta \cdot$$

$$\cdot \frac{\gamma\cdot(k+q)}{(k+q)^2 + i\varepsilon} \gamma_\nu$$ ; Now, $(q+k)^2 \simeq q^2 + 2k\cdot q =$

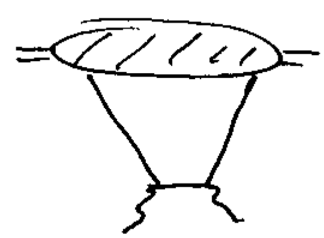
$= -Q^2 + 2k^+ q^-$

as $k^+$ is large. Similarly $(k+l+q)^2 \simeq -Q^2 +$

$+ 2q^-(k^+ + l^+)$

$$\Rightarrow \Gamma_{\mu\nu\beta} = \frac{1}{Q^4} \frac{\gamma_\mu\, \gamma\cdot(k+l+q)\, \gamma_\beta\, \gamma\cdot(k+q)\, \gamma_\nu}{\left(1 - \frac{2k^+q^-}{Q^2} - i\varepsilon\right)\left(1 - \frac{2(k^+ + l^+)q^-}{Q^2} - i\varepsilon\right)}$$

Seems like $\Gamma_{\mu\nu\beta} = o\left(\frac{1}{Q^4}\right)$, which is suppressed compared to $o\left(\frac{1}{Q^2}\right)$ diagram



However, when integrating over $dl^+$ can pick up the pole at $l^+ = -k^+ + \frac{Q^2}{2q^-}$, getting a $Q^2$ in the numerator.

$$\gamma\cdot(k+q) \simeq \gamma^+(k^-+q^-) + \gamma^-(k^++q^+) - \underline{\gamma}\cdot(\underline{k}+\underline{q})$$

$$\gamma\cdot(k+l+q) = \gamma^+(k^-+l^-+q^-) + \gamma^-(k^++l^++q^+) - \underline{\gamma}\cdot(\underline{k}+\underline{l}+\underline{q})$$

$\Rightarrow$ When taking Im part get $2k^+q^- = Q^2 \Rightarrow$

$\Rightarrow q^- = \dfrac{Q^2}{2k^+}$ $\Rightarrow$ 2nd denominator becomes

$Q^2$-independent: $1 - \dfrac{2(k^++l^+)q^-}{Q^2} = 1 - \dfrac{k^++l^+}{k^+} = -\dfrac{l^+}{k^+}$.

$\Rightarrow$ keep only $q^-$ terms in the numerator ($Q^2$-dep)

$\Rightarrow \gamma\cdot(k+q) \simeq \gamma^+ q^-$, $\gamma\cdot(k+l+q) \simeq \gamma^+ q^-$

$\Rightarrow \Gamma_{\mu\nu\beta} \sim \gamma^\mu\gamma^+\gamma^\beta\gamma^+\gamma^\nu$

$\Rightarrow \gamma^+\gamma^\beta\gamma^+$ is $\neq$ only if $\beta =$ "$-$" as $(\gamma^+)^2 =$

$$= \left(\frac{\gamma^0+\gamma^3}{\sqrt{2}}\right)^2 = \frac{1}{2}\left((\gamma^0)^2 + (\gamma^3)^2 + \overbrace{\{\gamma^0,\gamma^3\}}^{=0}\right) = \frac{1}{2}(1-1) = 0.$$

But if $\beta = -$ $\Rightarrow$ need $D_{\alpha+}(l)$

$$D_{\alpha\beta}(l) = \frac{-i}{l^2}\left[g_{\alpha\beta} - \frac{\eta_\alpha l_\beta + \eta_\beta l_\alpha}{\eta\cdot l}\right]$$

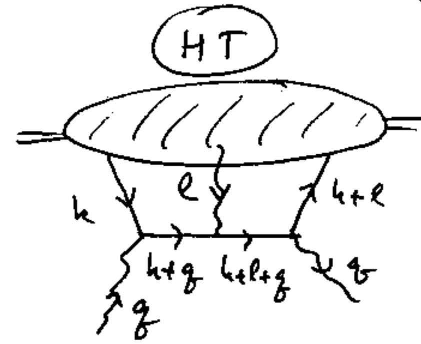
$D_{\alpha+} = \dfrac{-i}{l^2}\left[g_{\alpha+} - \dfrac{\eta_\alpha l_+}{l_+}\right] = 0$ $\Rightarrow$ never get $Q^2$

in the numerator $\Rightarrow$ $O\left(\dfrac{1}{Q^4}\right)$ "Higher Twist"

Let us work in Light Cone gauge defined by

$$\eta \cdot A = A^+ = 0$$

$$(\eta^+ = 0,\ \eta^- = 1,\ \underline{\eta} = 0)$$



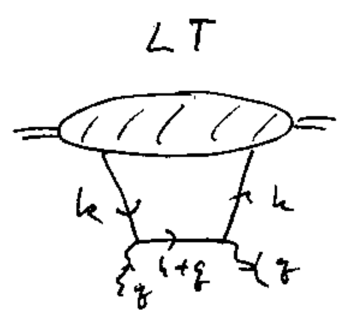

In DIS $Q^2$ is very large such that

$$\frac{|k^2|}{Q^2} \ll 1, \quad \frac{|l^2|}{Q^2} \ll 1$$

$\Rightarrow$ approximate $(k+q)^2 \simeq q^2 \simeq -Q^2$

$$(k+l+q)^2 \simeq q^2 \simeq -Q^2$$

$\Rightarrow$ diagram HT $\sim \frac{1}{Q^4}$




Compare with leading parton model diagram LT $\sim \frac{1}{Q^2}$

$\Rightarrow$ HT $\ll$ LT as for large $Q^2$: $\frac{1}{Q^4} \ll \frac{1}{Q^2}$.

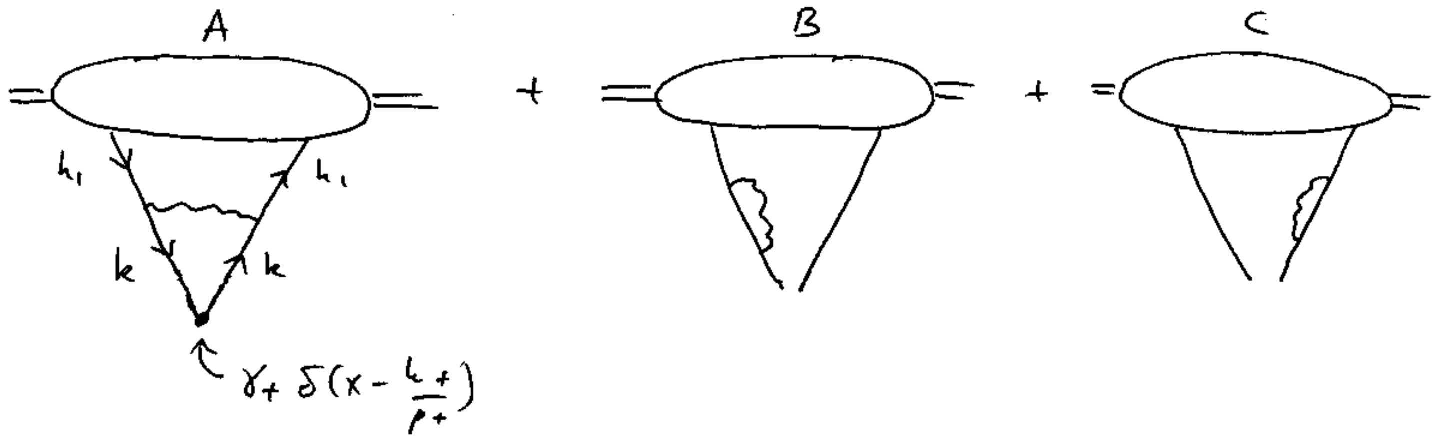
HT stand for Higher Twist' $\sim 1/Q^4$

LT —"— Leading Twist $\sim 1/Q^2$

$\Rightarrow$ Multiple rescatterings are <u>Higher</u> <u>Twists</u>, usually suppressed by $\Lambda^2/Q^2$ ($\Lambda$ - some small scale)

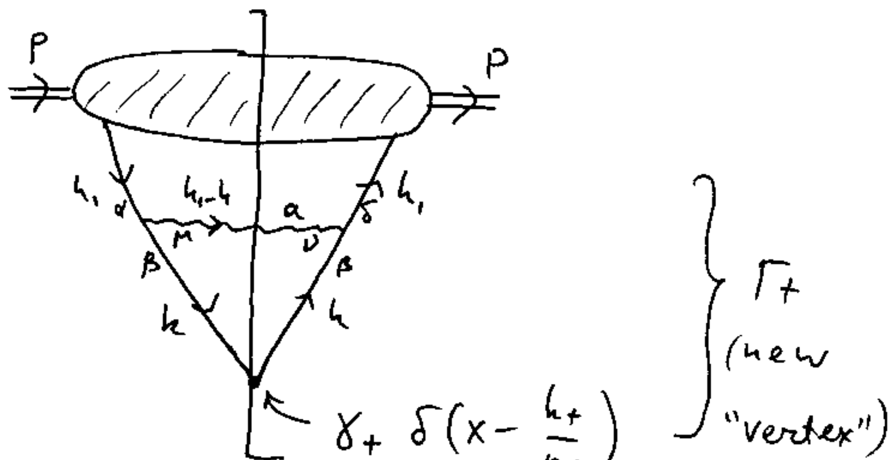
(true in LC gauge only!)

We need to calculate the following corrections to the parton model:




We will work out diagram A only: $|\underline{k}| \gg |\underline{k}_1|$

$$p^f(x, Q^2) = \frac{(2\pi)^3 E_p}{p_+}$$




$$(T^a T^a)_{\delta\alpha} = \delta_{\alpha\delta} \frac{N_c^2 - 1}{2N_c}$$

$$\Gamma_+ = (ig)^2 \overbrace{(T^a)_{\delta\beta} (T^a)_{\beta\alpha}} \int \frac{d^4k}{(2\pi)^4} \gamma_\nu \frac{i\gamma\cdot k}{k^2} \gamma_+ \frac{i\gamma\cdot k}{k^2} \gamma_\alpha \cdot$$

$$\cdot \delta\left(x - \frac{k_+}{p_+}\right) (-2\pi) \delta((k_1 - k)^2) \left[ g_{\mu\nu} - \frac{\eta_\mu (k_{1\nu} - k_\nu) + \eta_\nu (k_{1\mu} - k_\mu)}{k_{1+} - k_+} \right]$$

where we used the fact that gluon propagator in the $\eta \cdot A = A_+ = 0$ light cone gauge is

$$D_{\mu\nu}(l) = \frac{-i}{l^2} \left[ g_{\mu\nu} - \frac{\eta_\mu l_\nu + \eta_\nu l_\mu}{\eta \cdot l} \right] \text{ with } \frac{-i}{l^2} \to -2\pi \delta(l^2)$$

and $\eta \cdot l = l_+$.

First integrate over $k_-$:

$$\int dk_-\ \delta((k_1-k)^2) = \frac{1}{2(k_1-k)_+} \quad \text{with} \quad k_- = k_{1-} - \frac{(\underline{k}_1-\underline{k})^2}{2(k_1-k)_+}$$

Also, $k_+$-integration is easy: $\int dk_+\ \delta(x - \frac{k_+}{p_+}) = p_+$

Defining $\alpha_s \equiv \frac{g^2}{4\pi}$ we write $\left(C_F \equiv \frac{N_c^2-1}{2N_c}\right)$

$$\Pi_+ = -\frac{\alpha_s C_F}{4\pi^2}\delta_{\alpha\delta}\int \frac{d^2k}{k^4}\ \gamma_\nu\,\gamma\cdot k\,\gamma_+\,\gamma\cdot k\,\gamma_\mu\left[g_{\mu\nu} - \right.$$
$$\left. - \frac{\eta_\mu(k_{1\nu}-k_\nu)+\eta_\nu(k_{1\mu}-k_\mu)}{k_{1+}-k_+}\right]\frac{p_+}{(k_1-k)_+}$$

---

Evaluate $k^2$: $k^2 = 2k_+k_- - \underline{k}^2 = 2k_+\left(k_{1-} - \frac{(\underline{k}_1-\underline{k})^2}{2(k_1-k)_+}\right) - \underline{k}^2 =$

$= \Big|$ define $z = \frac{k_+}{k_{1+}}$ $= 2z\,k_{1+}k_{1-} - \frac{z}{1-z}(\underline{k}_1-\underline{k})^2 - \underline{k}^2 =$

$= z\,k_1^2 + z\,\underline{k}_1^2 - \frac{z}{1-z}\left(\underline{k}_1^2 - 2\underline{k}_1\cdot\underline{k} + \underline{k}^2\right) - \underline{k}^2 =$

$= z\,k_1^2 - \frac{1}{1-z}(\underline{k} - z\underline{k}_1)^2$

---

Evaluate $\gamma_\nu\,\gamma\cdot k\,\gamma_+\,\gamma\cdot k\,\gamma_\mu\left[g_{\mu\nu} - \frac{\eta_\mu(k_1-k)_\nu+\eta_\nu(k_1-k)_\mu}{k_{1+}-k_+}\right]$

First note that as $\{\gamma_\rho,\gamma_\sigma\} = 2g_{\rho\sigma}$

$$\gamma\cdot k\,\gamma_+\,\gamma\cdot k = \gamma\cdot k\Big[\underbrace{\{\gamma_+,\gamma\cdot k\}}_{2k_+} - \gamma\cdot k\,\gamma_+\Big] = \underset{(2)}{2k_+\,\gamma\cdot k} - \underset{(1)}{k^2\gamma_+}$$

Let's put ① and ② back into the monster expression:

$$① = -k^2\,\gamma_\nu\gamma_+\gamma_\mu\left[g_{\mu\nu} - \frac{\cancel{\eta_\mu}(k_1-k)_\nu + \cancel{\eta_\nu}(k_1-k)_\mu}{\eta\cdot(k_1-k)}\right] =$$

$$= \left(\text{as } \gamma_+^2 = 0\right) = -k^2\gamma_\mu\,\gamma_+\gamma^\mu = 2\gamma_+ k^2$$

since $\gamma_\mu\gamma_\alpha\gamma^\mu = -2\gamma_\alpha$

$$② = 2k_+\;\gamma_\nu\,\gamma\cdot k\,\gamma_\mu\left[g_{\mu\nu} - \frac{\eta_\mu(k_1-k)_\nu + \eta_\nu(k_1-k)_\mu}{\eta\cdot(k_1-k)}\right] =$$

$$= 2k_+\Big[\gamma_\mu\,\gamma\cdot k\,\gamma^\mu - \frac{1}{\eta\cdot(k_1-k)}\Big(\gamma\cdot(k_1-k)\,\gamma\cdot k\,\gamma_+ +$$

$$\gamma_+\,\gamma\cdot k\,\gamma\cdot(k_1-k)\Big)\Big] = 2k_+\Big[-2\,\gamma\cdot k - \frac{1}{\eta\cdot(k_1-k)}\cdot$$

$$\cdot\Big(-2k^2\gamma_+ + \gamma\cdot k_1\,\gamma\cdot k\,\gamma_+ + \gamma_+\,\gamma\cdot k\,\gamma\cdot k_1\Big)\Big]$$

we want to swap and move over here.

$$② = 2k_+\Big[-2\,\gamma\cdot k - \frac{1}{\eta\cdot(k_1-k)}\Big(-2k^2\gamma_+ + 2k\cdot k_1\,\gamma_+ +$$

$$+ 2k_+\,\gamma\cdot k_1 - 2k_{1+}\,\gamma\cdot k\Big)\Big]$$

Now we are interested in the regime where

$|\underline{k}| \gg |\underline{k}_1|$, $\underline{k}^2 \gg k_1^2$, i.e. $|\underline{k}|$ is VERY LARGE.

at the leading order in $|\underline{k}|$ : $k_- \approx -\dfrac{\underline{k}^2}{2k_{1+}(1-z)}$

For large $|\underline{k}|$: $\quad ① \simeq -2\gamma_+ \underline{k}^2 \frac{1}{1-z}$

as $k^2 = z k_1^2 - \frac{1}{1-z}(\underline{k} - z\underline{k}_1)^2 \to -\frac{\underline{k}^2}{1-z}$

$-2k^{*}k_1^{+}\gamma^{+}$
ss

$$② \simeq 2z\left[+\cancel{2}\gamma_+ \frac{\underline{k}^2}{\cancel{2}(1-z)} - \frac{1}{1-z}\left(\frac{2\underline{k}^2}{1-z}\gamma_+ - \cancel{\frac{\underline{k}^2}{1-z}\gamma_+} + \right.\right.$$

$$\left.\left. + \underbrace{\cancel{\frac{\underline{k}^2}{1-z}\gamma_+}}_{\simeq -2k_1^+\gamma^+k^-}\right)\right] = 2z\,\gamma_+ \frac{\underline{k}^2}{1-z}\left[1 - \frac{2}{1-z}\right] =$$

$$= -2\gamma_+ \underline{k}^2 \frac{z(1+z)}{(1-z)^2}$$

$$\Rightarrow \boxed{① + ② = -2\gamma_+ \underline{k}^2 \frac{1+z^2}{(1-z)^2}}$$

Plugging it all back we get

$$\Gamma_+ = -\frac{\alpha_s C_F}{4\pi^2}\delta_{\alpha\delta}\int \underbrace{\frac{d^2k}{\underline{k}^4}(1-z)^2}_{(1/\underline{k}^2)^2}(-2)\gamma_+ \underline{k}^2 \frac{1+z^2}{(1-z)^2}\,\frac{p_+/k_{1+}}{1-z}$$

$\Rightarrow$ defining Bjorken (or Feynman) $x$ for quark $k_1$

as $x_1 \equiv \frac{k_{1+}}{p_+}$ we get

$$\boxed{\Gamma_+ = \gamma_+ \frac{1}{x_1}\frac{\alpha_s C_F}{2\pi}\int \frac{d\underline{k}^2}{\underline{k}^2}\frac{1+z^2}{1-z}}$$

$$\Gamma_+ = \gamma_+ \frac{1}{x_1} \frac{\alpha_s C_F}{2\pi} \int_{\underline{k}_1^2}^{Q^2} \frac{d\underline{k}^2}{\underline{k}^2} \frac{1+z^2}{1-z}$$

$$\Gamma_+ \sim \alpha_s \cdot \ln\left(Q^2/\underline{k}_1^2\right) \sim \alpha_s \ln Q^2$$

$\alpha_s << 1$ (perturbation theory, small coupling)

$\ln Q^2 >> 1$ ( DIS with large $Q^2$)

$\alpha \ln Q^2 \sim 1$ our resummation parameter!

"Leading Logarithmic Approximation"

Remember: we neglected terms suppressed by $\frac{\underline{k}_1^2}{\underline{k}^2}, \frac{\underline{k}_1^4}{\underline{k}^4}, \ldots \Rightarrow$ they give

$$\int_{\underline{k}_1^2}^{Q^2} \frac{dk^2}{k^4} \underline{k}_1^2 \sim \left(\frac{1}{\underline{k}_1^2} - \frac{1}{Q^2}\right) \underline{k}_1^2 \sim \underbrace{1}_{\text{no log}} - \underbrace{\frac{\underline{k}_1^2}{Q^2}}_{\text{higher twist}}$$

Old (LO) Parton Model vertex (Mueller vertex) was $\gamma_+ \delta\left(x - \frac{k_+}{p_+}\right)$ ~ same $\gamma_+$ matrix as $\Gamma_+$

$$\Rightarrow \boxed{Q^2 \frac{\partial}{\partial Q^2} q^f(x, Q^2) = \frac{\alpha}{2\pi} \int_x^1 \frac{dx_1}{x_1} \gamma_{qq}\left(\frac{x}{x_1}\right) q^f(x_1, Q^2)}$$