Last time: derived DGLAP evolution equation:

$$\Sigma(x,Q^2) = \sum_f \left[q^f(x,Q^2) + q^{\bar f}(x,Q^2)\right] \sim \text{flavor singlet}$$

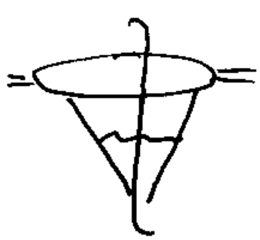
$$\Delta_{ff'}(x,Q^2) = q^f(x,Q^2) - q^{f'}(x,Q^2) \sim \text{flavor non-singlet}$$

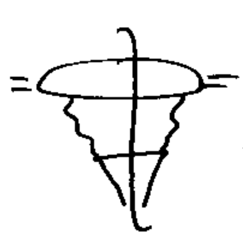
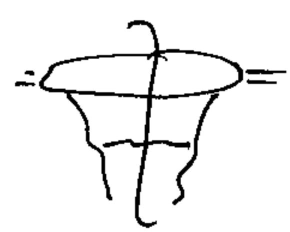
$G(x,Q^2) \sim$ gluon distribution 

$$\boxed{Q^2 \frac{\partial}{\partial Q^2} \Delta_{ff'}(x,Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{dx_1}{x_1}\, \gamma_{qq}\left(\frac{x}{x_1}\right) \Delta_{ff'}(x_1,Q^2)}$$

$$\boxed{Q^2 \frac{\partial}{\partial Q^2} \begin{pmatrix} \Sigma(x,Q^2) \\ G(x,Q^2) \end{pmatrix} = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{dx_1}{x_1} \begin{pmatrix} \gamma_{qq}\left(\frac{x}{x_1}\right) & \gamma_{qG}\left(\frac{x}{x_1}\right) \\ \gamma_{Gq}\left(\frac{x}{x_1}\right) & \gamma_{GG}\left(\frac{x}{x_1}\right) \end{pmatrix} \cdot \begin{pmatrix} \Sigma(x_1,Q^2) \\ G(x_1,Q^2) \end{pmatrix}}$$

$\gamma_{qq}(z) = C_F \left(\frac{1+z^2}{1-z}\right)_+$; $\gamma_{Gq}(z) = C_F \frac{1+(1-z)^2}{z}$;

$\gamma_{qG}(z) = N_f\left[z^2 + (1-z)^2\right]$  ; $\gamma_{GG}(z) =$ 

$$\gamma_{GG}(z) = 2N_c\left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\right] + \frac{11N_c - 2N_f}{6}\,\delta(z-1).$$

~ resums powers of $\alpha_s \ln \frac{Q^2}{\Lambda^2}$ ~ leading logarithmic approximation (LLA).

~ "+" notation:

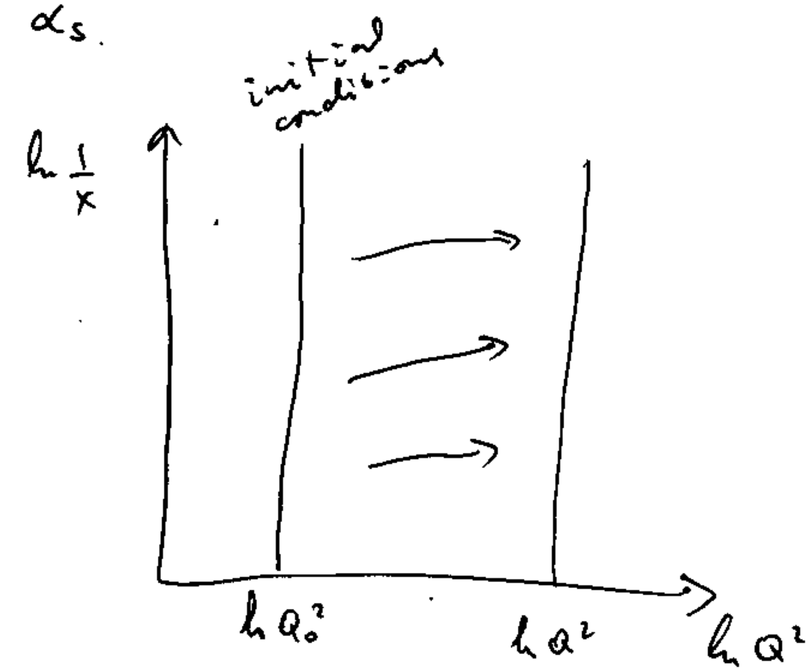
$$\int_0^1 dz\,[h(z)]_+\, f(z) = \int_0^1 dz\, h(z)\,[f(z) - f(1)].$$

(needed to include virtual corrections)

(NB) To solve DGLAP need initial conditions, which provide non-perturbative (non-calculable) input!

(NB) $\alpha_s = \alpha_s(Q^2)$ ~ it's not s, but Q^2 which sets the scale for α_s.

How DGLAP works:



DGLAP at small-x.

Gluons dominate at small-x (see attached plot) $\Rightarrow$ forget about quarks for now. Evolution for $x\,G$ is

$$Q^2 \frac{\partial}{\partial Q^2}\, G(x, Q^2) = \frac{\alpha(Q^2)}{2\pi} \int_x^1 \frac{dx'}{x'}\, \gamma_{GG}\left(\frac{x}{x'}\right) G(x', Q^2)$$

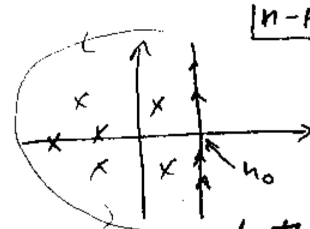
where $\gamma_{GG}(z) = 2N_c \left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \right] + \frac{11 N_c - 2N_f}{6}\, \delta(z-1)$

Def. $\approx \frac{2N_c}{z}$ at small z!

Consider **moments** of $x\,G(x, Q^2)$:

n-plane

$$G_n(Q^2) \equiv \int_0^1 dx\; x^{n-1}\, G(x, Q^2) \quad \begin{pmatrix}\text{Mellin} \\ \text{transform}\end{pmatrix}$$



such that $\displaystyle G(x, Q^2) = \int \frac{dn}{2\pi i}\, x^{-n}\, G_n(Q^2)$ to the right of all singularities

$$\left(\underline{\text{Check}}: \int \frac{dn}{2\pi i}\, x^{-n} \cdot (x')^{n-1} = \frac{1}{x'} \int \frac{dn}{2\pi i}\, e^{n \ln(x'/x)} = \Big|\, n = i\lambda + n_0 \right.$$

$$\left. = \frac{1}{x'}\, e^{n_0 \ln(x'/x)} \underbrace{\int_{-\infty}^{\infty} \frac{d\lambda}{2\pi}\, e^{i\lambda \ln(x'/x)}}_{\delta\left(\ln \frac{x'}{x}\right)} = \delta(x' - x) \right)$$

Multiply evolution equation for $G(x, Q^2)$ by x^{n-1} and integrate over x from 0 to 1:

$$Q^2 \frac{\partial}{\partial Q^2}\, G_n(Q^2) = \frac{\alpha(Q^2)}{2\pi} \int_0^1 dx\; x^{n-1} \int_x^1 \frac{dx'}{x'}\, \gamma_{GG}\left(\frac{x}{x'}\right) G(x', Q^2)$$

$$= \frac{\alpha(Q^2)}{2\pi} \int_0^1 dx' \, (x')^{n-1} \, G(x', Q^2) \cdot \int_0^{x'} \frac{dx}{x'} \left(\frac{x}{x'}\right)^{n-1} \gamma_{GG}\left(\frac{x}{x'}\right) = \Big| z = \frac{x}{x'}$$

$$= \frac{\alpha(Q^2)}{2\pi} \underbrace{\int_0^1 dx' \, (x')^{n-1} \, G(x', Q^2)}_{G_n(Q^2)} \cdot \underbrace{\int_0^1 dz \cdot z^{n-1} \gamma_{GG}(z)}_{\gamma_{GG}^{(n)}}$$

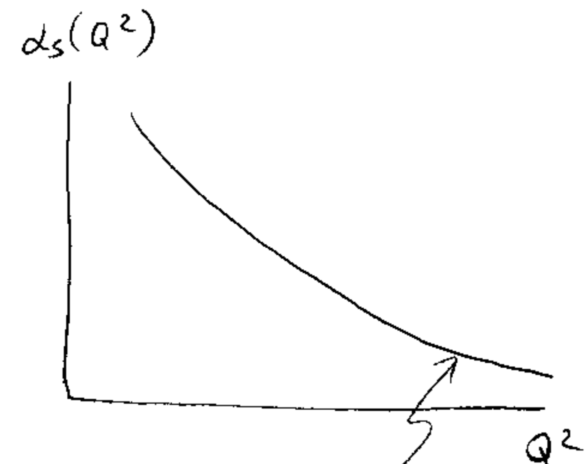
$$\boxed{Q^2 \frac{\partial}{\partial Q^2} G_n(Q^2) = \frac{\alpha(Q^2)}{2\pi} \gamma_{GG}^{(n)} \, G_n(Q^2)}$$

DGLAP in Mellin space

Solution: $$G_n(Q^2) = e^{\int_{Q_0^2}^{Q^2} \frac{dQ'^2}{Q'^2} \cdot \frac{\alpha(Q'^2)}{2\pi} \gamma_{GG}^{(n)}} \, G_n(Q_0^2).$$

Running coupling case

$$\alpha(Q^2) = \frac{1}{\beta_2 \ln Q^2/\Lambda^2} \quad \text{with} \quad \beta_2 = \frac{11 N_c - 2 N_f}{12\pi}$$



Gross, Wilczek & Politzer

Nobel Prize of 2004

coupling is small at large Q^2 (short transverse distances $x_\perp \sim \frac{1}{Q}$) $\Rightarrow$ asymptotic freedom!

$$\int_{Q_0^2}^{Q^2} \frac{dQ'^2}{Q'^2} \frac{\alpha(Q'^2)}{2\pi} = \frac{1}{2\pi\beta_2} \int_{Q_0^2}^{Q^2} \frac{dQ'^2}{Q'^2} \frac{1}{\ln Q'^2/\Lambda^2} =$$

$$= \frac{1}{2\pi\beta_2} \int_{\ln Q_0^2/\Lambda^2}^{\ln Q^2/\Lambda^2} d\ln Q'^2/\Lambda^2 \, \frac{1}{\ln Q'^2/\Lambda^2} = \frac{1}{2\pi\beta_2} \ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right).$$

$$\Rightarrow \quad G_n(Q^2) = e^{\frac{\gamma_{GG}(n)}{2\pi\beta_2} \ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)} G_n(Q_0^2) \quad \Rightarrow$$

$$G(x, Q^2) = \int \frac{dn}{2\pi i} x^{-n} e^{\frac{\gamma_{GG}^{(n)}}{2\pi\beta_2} \ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)} G_n(Q_0^2).$$

at small-x: $\gamma_{GG}(z) \approx \frac{2N_c}{z}$

$$\Rightarrow \gamma_{GG}^{(n)} \simeq \int_0^1 dz \cdot z^{n-2} \, 2N_c \overset{\text{for } n>1}{=} \frac{2N_c}{n-1}$$

Evaluate the integral over n in the saddle point (a.k.a. stationary phase) approximation:

$$G(x, Q^2) = \int \frac{dn}{2\pi i} e^{n \ln\frac{1}{x} + \frac{N_c}{n-1} \frac{1}{\pi\beta_2} \ln\left(\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right)} G_n(Q_0^2)$$

Assume that most n-dependence is in the exponent. At small-x $\ln\frac{1}{x}$ is very large $\Rightarrow$

$\Rightarrow$ the exponent oscillates wildly as n varies.

Oscillations are not there only at the saddle point :

$$\frac{d}{dn}\left[n \ln\frac{1}{x} + \frac{N_c}{n-1}\frac{1}{\pi\beta_2}\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)\right]\Bigg|_{n=n_0} = 0$$

$$\ln\frac{1}{x} - \frac{N_c}{(n_0-1)^2}\frac{1}{\pi\beta_2}\ln\left(\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right) = 0$$

$$n_0 - 1 = \pm\sqrt{\frac{N_c}{\pi\beta_2}\ln\left(\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right)\frac{1}{\ln\frac{1}{x}}}$$

"+" dominates (gives larger contribution).
to $(n_0-1)\ln\frac{1}{x}$

To estimate the integral we define the power of the exponent

$$P(n) = n\ln\frac{1}{x} + \frac{N_c}{n-1}\frac{1}{\pi\beta_2}\ln\left(\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right)$$

and expand

$$P(n) \approx P(n_0) + \frac{1}{2}(n-n_0)^2 P''(n_0)$$

where $$P''(n_0) = +\frac{2N_c}{(n_0-1)^3}\frac{1}{\pi\beta_2}\ln\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2} = \frac{2N_c}{\pi\beta_2}\ln\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\cdot$$

$$\left(\frac{\pi b}{N_c}\right)^{3/2}\left[\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)\right]^{-3/2}\ln^{3/2}\frac{1}{x} = 2\left(\frac{\pi\beta_2}{N_c}\right)^{1/2}\ln^{3/2}\frac{1}{x}\cdot\left[\ln\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right]^{-1/2}.$$

$$P(n_0) = \ln\frac{1}{x} + 2\sqrt{\frac{N_c}{\pi\beta_2}\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)\ln\frac{1}{x}}.$$

$$\int \frac{dn}{2\pi i}\, e^{P(n_0) + \frac{1}{2}(n-n_0)^2 P''(n_0)} = \Big|\, n - n_0 = i\xi \; = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi}\, e^{P(n_0) - \frac{1}{2}\xi^2 P''(n_0)} =$$

$$= \frac{1}{2\pi}\, e^{P(n_0)} \sqrt{\frac{2\pi}{P''(n_0)}} = \frac{e^{P(n_0)}}{\sqrt{2\pi P''(n_0)}}$$

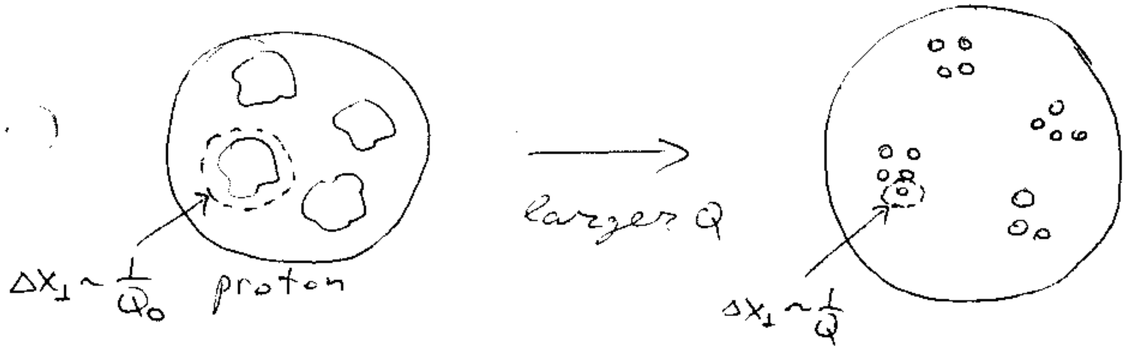
we obtain

$$\boxed{x G(x, Q^2) = G_{n_0}(Q_0^2) \cdot e^{2\sqrt{\frac{N_c}{\pi \beta_2} \ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right) \ln\frac{1}{x}}} \cdot \frac{1}{\sqrt{4\pi}} \cdot \left(\frac{N_c}{\pi\beta_2}\right)^{1/4} \ln^{-3/4}\frac{1}{x} \left[\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)\right]^{1/4}}$$

Therefore, $xG \sim e^{2\sqrt{\frac{N_c}{\pi\beta_2} \ln\frac{1}{x} \ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)}}$

xG grows at small-x, slower than a power of x but faster than any power of $\ln\frac{1}{x}$. $\Rightarrow$ may explain rise of xG at small-x...

How DGLAP works: we increase Q/resolution, see more partons



Renormalization Groups.