Last time: solved DGLAP equation for $G(x, Q^2)$

at small-x:

$$\boxed{Q^2 \frac{\partial}{\partial Q^2} G(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \int\limits_x^1 \frac{dx_1}{x_1} \gamma_{GG}\left(\frac{x}{x_1}\right) G(x_1, Q^2)}$$

We went to Mellin moment space:

$$G_n(Q^2) = \int\limits_0^1 dx \cdot x^{n-1} \, G(x, Q^2)$$

There the eqn. became:

$$\boxed{Q^2 \frac{\partial}{\partial Q^2} G_n(Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \gamma_{GG}^{(n)} \, G_n(Q^2)}$$

This we can easily solve

$$\boxed{G_n(Q^2) = G_n(Q_0^2) \exp\left\{ \int\limits_{Q_0^2}^{Q^2} \frac{d\mu^2}{\mu^2} \frac{\alpha_s(\mu^2)}{2\pi} \gamma_{GG}^{(n)} \right\}}$$

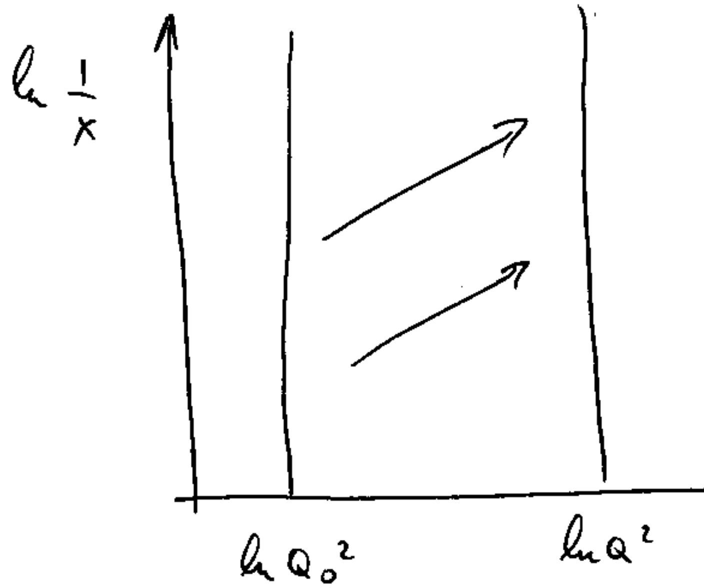
At small-z $\quad \gamma_{GG}(z) \simeq \frac{2N_c}{z} \Rightarrow \gamma_{GG}^{(n)} = \frac{2N_c}{n-1} \Rightarrow$ got

$$\boxed{x G(x, Q^2) = \int \frac{dn}{2\pi i} \, e^{(n-1)\ln\frac{1}{x} + \frac{1}{n-1}\frac{N_c}{\pi\beta_2}\ln\left[\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right]} \, G_n(Q_0^2)}$$

Then we did the integral using saddle point method, obtaining

$$x G \propto e^{2\sqrt{\frac{N_c}{\pi\beta_2}\ln\frac{1}{x}\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right)}}$$

$\Rightarrow$ xG grows as x gets small & Q^2 gets large!



DLA DGLAP moves PDF's toward higher Q^2 & lower-x.

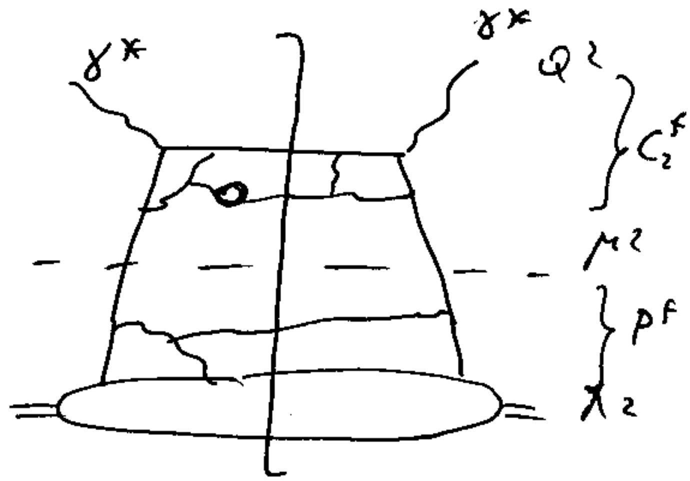
Particle Production in High Energy Hadronic Collisions

(cont'd)

Collinear Factorization

In DIS:

$$F_2(x,Q^2) = \sum_{f,\bar{f},\ \text{gluons}} \int_0^1 d\xi \; C_2^f\left(\frac{x}{\xi}, Q^2, \mu^2\right) \cdot p^f(\xi, \mu^2, \Lambda^2).$$



C_2 ~ coefficient function ~ perturbative

p^f ~ distribution function ~ non-perturbative

Collinear factorization in DIS is a <u>theorem</u> which can be <u>proven</u> $\Rightarrow$ must be right! (at large-Q^2 only!)

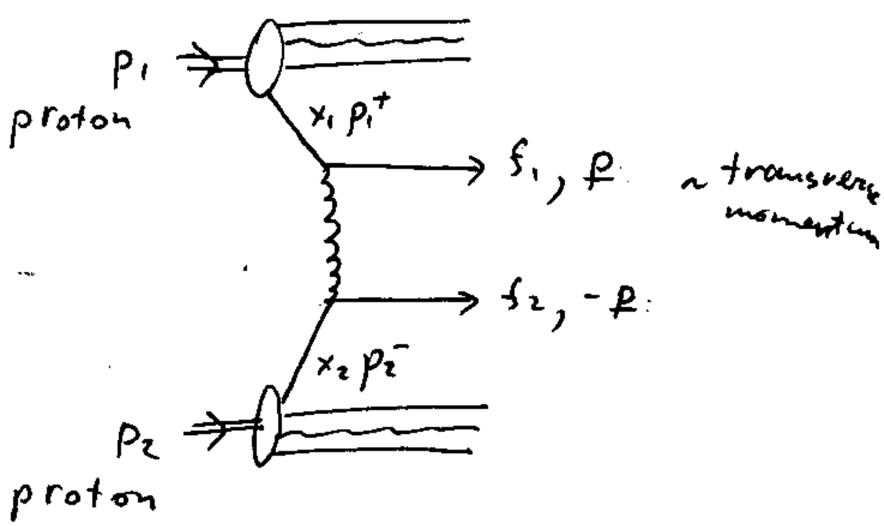
~ at LO have $C_2^f = \delta\left(\frac{x}{\xi} - 1\right) e_f^2$, f = quarks only

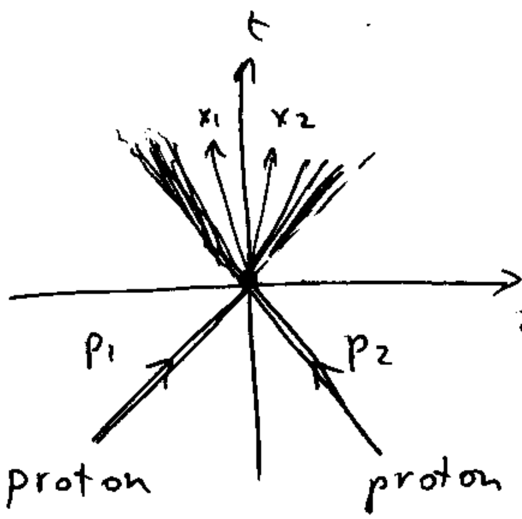
$$\Rightarrow F_2(x, Q^2) = \sum_f \underbrace{\int_0^1 d\xi \, \delta\left(\frac{x}{\xi} - 1\right)}_{x} e_f^2 \, q^f(\xi)$$

$$= \sum_f e_f^2 \, x \, q^f(x) \quad \text{as expected!}$$

<u>Jet Production in Hadronic Collisions.</u>

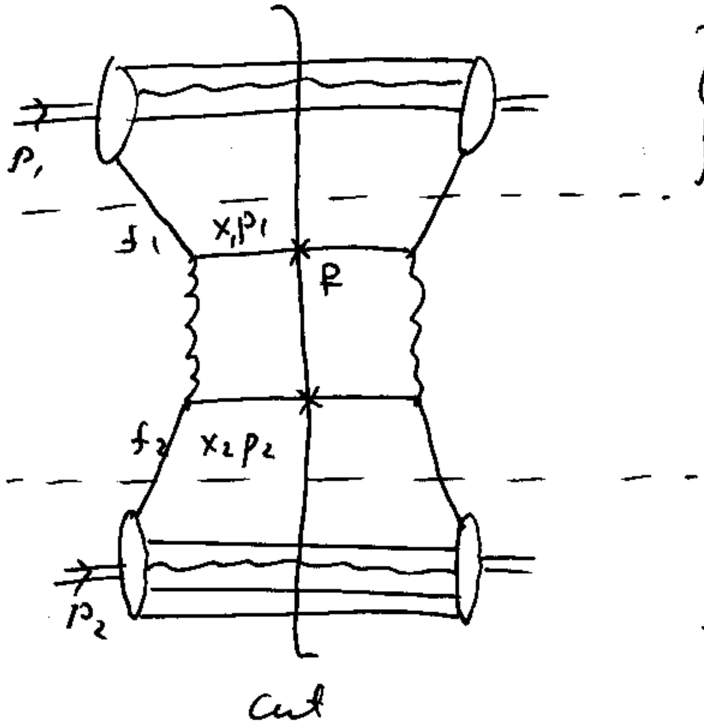
Collinear factorization also applies to hadron-hadron collisions. Consider quark production:





~ collision happens very fast on proton's time scales $\Rightarrow$ factorization.

Square the diagram:



$\}$ $p^{f_1}(x_1, \mu^2)$ ~ parton distribution

$\}$ $\mathcal{E}_p \dfrac{d\hat{\sigma}_{f_1 f_2 \to f_1' f_2'}}{d^3 p}$ ~ "hard" cross section

$\}$ $p^{f_2}(x_2, \mu^2)$ ~ parton distribution

The collinear factorization formula then reads:

$$\boxed{\mathcal{E}_p \frac{d\sigma}{d^3 p} = \sum_{i,j} \int_0^1 dx_1 \int_0^1 dx_2 \; p^{f_i}(x_1, \mu^2) \cdot \mathcal{E}_p \frac{d\hat{\sigma}_{f_i f_j \to f_i' f_j'}}{d^3 p} \cdot p^{f_j}(x_2, \mu^2)}$$

Usually put $\mu^2 = p_T^2$ for large p_T jets.

After the collision quarks (gluons) that are produced get dressed by further emissions. But the flow of energy is not likely to be modified much by those. (Still people construct other IR-safe observables insensitive to late-time emissions: (— + ... + ...))

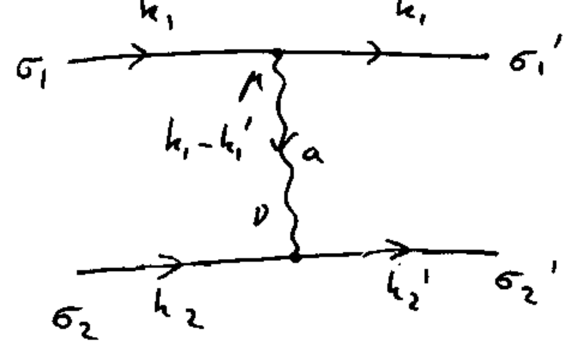
<u>Example</u> Quark jet production (coming from quarks). Replace $p^{f_i} \to q^f$ $\Rightarrow$ write

$$\varepsilon_p \frac{d\sigma}{d^3p} = \sum_{f_1,f_2} \int_0^1 dx_1\, dx_2\; q^{f_1}(x_1, p_T^2)\; \varepsilon_p \frac{d\hat{\sigma}_{f_1 f_2 \to f_1' f_2'}}{d^3p}\; q^{f_2}(x_2, p_T^2)$$

q^f ~ to be found from DGLAP (PDF data)

We can calculate the hard cross section:

$$d\hat{\sigma} = \frac{1}{\underbrace{2\varepsilon_1\, 2\varepsilon_2 \cdot 2 \cdot}_{|\vec{v}_1 - \vec{v}_2|}} \frac{d^3k_1'}{(2\pi)^3 2\varepsilon_1'} \cdot$$



$$\cdot \frac{d^3k_2'}{(2\pi)^3 2\varepsilon_2'} (2\pi)^4 \delta^{(4)}(k_1 + k_2 - k_1' - k_2') \cdot$$

$$\cdot |M|^2 \left[\delta^3(\vec{k}_1' - \vec{p}) + \delta^3(\vec{k}_2' - \vec{p}) \right] \cdot d^3p$$

measured jet can be either quark!

$$\Rightarrow \varepsilon_p \frac{d\hat{\sigma}}{d^3p} = ?$$

$\delta^{(3)}$ kills d^3k_2' kills d^3k_1'

$$\frac{d^3k_1'}{(2\pi)^3 2\varepsilon_1'} \frac{d^3k_2'}{(2\pi)^3 2\varepsilon_2'} (2\pi)^4 \delta^{(4)}(k_1 + k_2 - k_1' - k_2')\, \delta^3(\vec{k}_1' - \vec{p}) =$$

$$= \frac{1}{(2\pi)^2} \frac{1}{4\varepsilon_p \varepsilon_2'} \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_p - \varepsilon_2')$$

Define Mandelstam variables

$$\hat{s} \equiv (k_1 + k_2)^2 \; ; \quad \hat{t} \equiv (k_1 - k_1')^2 \; , \quad \hat{u} \equiv (k_1 - k_2')^2 .$$

assuming massless quarks & energy-momentum conservation write

$$\hat{s} = (k_1 + k_2)^2 = (k_1' + k_2')^2 = 2 k_1' \cdot k_2'$$

$$\hat{u} = (k_1 - k_2')^2 = -2 k_1 \cdot k_2'$$

$$\hat{t} = (k_1 - k_1')^2 = -2 k_1 \cdot k_1' = (k_2 - k_2')^2 = -2 k_2 \cdot k_2'$$

$$\hat{s} + \hat{t} + \hat{u} = 2 k_2' \cdot (k_1' - k_1 - k_2)$$

Now, remembering that we have imposed only 3-momentum conservation we write

$$(k_1' - k_1 - k_2)^\mu = - k_2'^\mu + (\varepsilon_1' - \varepsilon_1 - \varepsilon_2 + \varepsilon_2', \vec{0})$$

$$\Rightarrow \hat{s} + \hat{t} + \hat{u} = \underbrace{-2(k_2')^2}_{\overset{\shortparallel}{0} \text{ (massless quarks)}} + 2 \varepsilon_2' \cdot (\varepsilon_1' + \varepsilon_2' - \varepsilon_1 - \varepsilon_2)$$

$$\Rightarrow \boxed{\delta(\hat{s} + \hat{t} + \hat{u}) = \frac{1}{2\varepsilon_2'} \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_1' - \varepsilon_2')}$$

$$\Rightarrow \frac{d^3 k_1'}{(2\pi)^3 2\varepsilon_1'} \frac{d^3 k_2'}{(2\pi)^3 2\varepsilon_2'} (2\pi)^4 \delta^{(4)}(k_1 + k_2 - k_1' - k_2') \, \delta^3(\vec{k}_1' - \vec{p}) =$$

$$= \frac{1}{(2\pi)^2} \frac{1}{2\varepsilon_p \, 2\varepsilon_2'} \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_1' - \varepsilon_2') = \frac{1}{(2\pi)^2} \frac{1}{2\varepsilon_p} \delta(\hat{s} + \hat{t} + \hat{u}) .$$

$$\Rightarrow \boxed{E_p \frac{d\hat{\sigma}}{d^3p} = \frac{1}{8\,\varepsilon_1\varepsilon_2}\cdot\cdot\frac{1}{2(2\pi)^2}\,\delta(\hat{s}+\hat{t}+\hat{u})\cdot|M|^2}$$

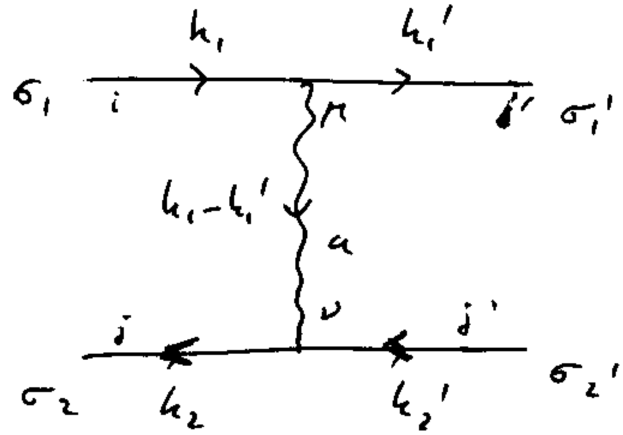
+ term with $\vec{k}_2' = \vec{p}$

Now let's find the amplitude squared:

$$M = (ig)^2\,\bar{u}_{\sigma_1'}(k_1')\gamma^\mu u_{\sigma_1}(k_1)\;\bar{u}_{\sigma_2'}(k_2')\gamma^\nu u_{\sigma_2}(k_2)\,\cdot$$

$$\cdot\,\frac{-i\,g_{\mu\nu}}{\underbrace{(k_1-k_2)^2}_{=\hat{t}}}\sum_a (T^a)_{i'i}\,(T^a)_{j'j}$$

see figure:



average over initial polarizations

average over initial colors

$$\frac{1}{4}\sum_{\substack{\sigma_1,\sigma_2\\ \sigma_1',\sigma_2'}}\frac{1}{N_c^2}\sum_{\substack{i,j\\ i',j'}}|M|^2 = \frac{1}{4}g^4\frac{1}{N_c^2}\cdot\underbrace{tr\left(T^aT^b\right)}_{\frac{1}{2}\delta^{ab}}\underbrace{tr\left(T^aT^b\right)}_{\frac{1}{2}\delta^{ab}}$$

$$\cdot\,\frac{1}{\hat{t}^2}\cdot tr\left[\not{k}_1'\gamma^\mu\not{k}_1\gamma^\rho\right]\,tr\left[\not{k}_2'\gamma^\mu\not{k}_2\gamma^\rho\right]$$

as $\sum_{\sigma_1,\sigma_1'}\bar{u}_{\sigma_1'}(k_1')\gamma^\mu u_{\sigma_1}(k_1)\left[\bar{u}_{\sigma_1'}(k_1')\gamma^\rho u_{\sigma_1}(k_1)\right]^\dagger$ = (see DIS discussion) $= \sum_{\sigma_1,\sigma_1'}\bar{u}_{\sigma_1'}(k_1')\gamma^\mu\underbrace{u_{\sigma_1}(k_1)\,\bar{u}_{\sigma_1}(k_1)}_{\not{k}_1\text{ after sum}}\gamma^\rho u_{\sigma_1'}(k_1')$

$= \mathrm{Tr}\left[\not{k}_1' \gamma^\mu k_1 \gamma^\rho \right]$ as advertised.

$$\underbrace{\frac{1}{4} \sum_{pol's} \frac{1}{N_c^2} \sum_{colors} |M|^2}_{\langle |M|^2 \rangle} = \frac{g^4}{\cancel{4} N_c^2} \cdot \frac{1}{\hat{t}^2} \cdot \frac{1}{\cancel{4}} \underbrace{\delta^{aa}}_{N_c^2 - 1} \cdot \cancel{4} \left(k_1^\rho k_1'^\mu + \right.$$

$$\left. + k_1^\mu k_1'^\rho - g^{\mu\rho} k_1 \cdot k_1' \right) \cancel{4} \left(k_2'^\mu k_2^\rho + k_2'^\rho k_2^\mu - g^{\mu\rho} k_2 \cdot k_2' \right)$$

$$= \frac{g^4}{\hat{t}^2} \frac{N_c^2 - 1}{N_c^2} \left(k_1^\rho k_1'^\mu + k_1^\mu k_1'^\rho - g^{\mu\rho} k_1 \cdot k_1' \right) \left(2 k_2'^\mu k_2^\rho - \right.$$

$$\left. - g^{\mu\rho} k_2 \cdot k_2' \right) = \frac{g^4}{\hat{t}^2} \frac{N_c^2 - 1}{N_c^2} \left[2 k_1' \cdot k_2' \, k_1 \cdot k_2 + 2 k_1 \cdot k_2' \right.$$

$$\left. k_2 \cdot k_1' - \cancel{2 k_2 \cdot k_2' \, k_1 \cdot k_1'} - \cancel{2 k_1 \cdot k_1' \, k_2 \cdot k_2'} + \cancel{4 k_1 \cdot k_1' \, k_2 \cdot k_2} \right]$$

$$= 2 \frac{g^4}{\hat{t}^2} \frac{N_c^2 - 1}{N_c^2} \left[\underbrace{k_1' \cdot k_2'}_{\hat{s}/2} \underbrace{k_1 \cdot k_2}_{\hat{s}/2} + \underbrace{k_1 \cdot k_2'}_{-\frac{\hat{u}}{2}} \underbrace{k_2 \cdot k_1'}_{-\frac{\hat{u}}{2}} \right]$$

$$= \frac{1}{2} \frac{g^4}{\hat{t}^2} \frac{N_c^2 - 1}{N_c^2} \left[\hat{s}^2 + \hat{u}^2 \right]$$

Finally we plug this back into the expression for x-section:

$$E_p \frac{d\hat{\sigma}}{d^3 p} = \frac{1}{8 E_1 E_2} \frac{1}{2 (2\pi)^2} \delta(\hat{s} + \hat{t} + \hat{u}) \frac{1}{2} \frac{g^4}{\hat{t}^2} \frac{N_c^2 - 1}{N_c^2} \left[\hat{s}^2 + \hat{u}^2 \right]$$

Remember $C_F = \frac{N_c^2 - 1}{2 N_c}$ (fundamental Casimir)

Also pick $k_1 = (E_1, 0, 0, E_1) \Rightarrow k_2 = (E_2, 0, 0, -E_2) \Rightarrow$

$$\hat{s} = 2k_1 \cdot k_2 = 4\,\varepsilon_1 \varepsilon_2$$

$$\Rightarrow \varepsilon_p \frac{d\hat{\sigma}}{d^3 p} = \frac{1}{\hat{s}}\, \alpha_s^2\, \delta(\hat{s}+\hat{t}+\hat{u})\, \frac{1}{\hat{t}^2}\, \frac{C_F}{N_c} \left[\hat{s}^2 + \hat{u}^2\right]$$

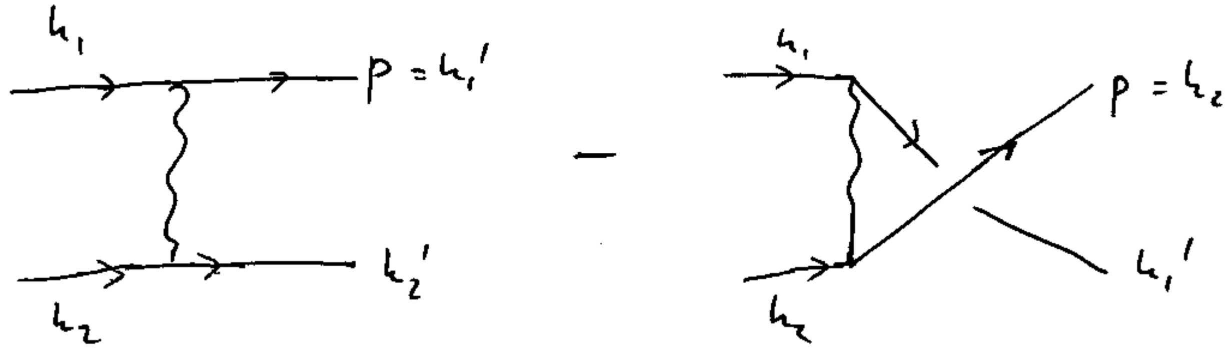
$$\Rightarrow \boxed{\varepsilon_p \frac{d\hat{\sigma}}{d^3 p} = \alpha_s^2\, \frac{C_F}{N_c}\, \frac{1}{\hat{s}\,\hat{t}^2} \left[\hat{s}^2 + \hat{u}^2\right] \delta(\hat{s}+\hat{t}+\hat{u})}$$

+ term with $\vec{k}_2' = \vec{p}$

$$C_F = \frac{N_c^2 - 1}{2N_c} = \frac{8}{6} = \frac{4}{3}$$

$$\Rightarrow \varepsilon_p \frac{d\hat{\sigma}}{d^3 p} = 2 \cdot \frac{4}{9}\, \alpha_s^2\, \frac{1}{\hat{s}\,\hat{t}^2} \left[\hat{s}^2 + \hat{u}^2\right] \delta(\hat{s}+\hat{t}+\hat{u}) + \ldots$$

However, one has to be careful, we forgot the term with $\hat{k}_2' = \vec{p}$. Diagrammatically we have



$\hat{t}_{old} = (k_1 - p)^2 = (k_1 - k_1')^2$

$\hat{u}_{old} = (k_1 - k_2')^2$

here $\hat{t}_{New} = (k_1 - p)^2 = (k_1 - k_2')^2 = \hat{u}_{old}$

$\hat{u}_{New} = (k_1 - k_1')^2 = \hat{t}_{old}$

$\Rightarrow$ the second graph squared is obtained by replacing

$$\hat{t} \leftrightarrow \hat{u}.$$

The final answer reads:

$$E_p \frac{d\hat{\sigma}_{qq\to qq}}{d^3p} = \alpha_s^2 \frac{C_F}{N_c} \frac{1}{\hat{s}} \delta(\hat{s}+\hat{t}+\hat{u}) \left[\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2} + \frac{\hat{s}^2+\hat{t}^2}{\hat{u}^2} \underbrace{- \frac{2}{N_c} \frac{\hat{s}^2}{\hat{t}\,\hat{u}}} \right]$$

interference term we did not calculate. (see next page)

To find jet production x-section need to calculate hard cross section $\hat{\sigma}$ & convolute it with 2 PDF's.

If quarks are distinguishable => no crossing term, no interference =>

$$E_p \frac{d\hat{\sigma}_{ud\to ud}}{d^3p} = \alpha_s^2 \frac{C_F}{N_c} \frac{1}{\hat{s}} \delta(\hat{s}+\hat{t}+\hat{u}) \frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}.$$