Axial Anomaly

Consider massless QED as an example:

$$\mathcal{L} = \bar{\psi}\, i\gamma\cdot\partial\psi - e\bar{\psi}\gamma^\mu A_\mu \psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$\psi \sim$ electron field, $A_\mu \sim$ photon field.

$\mathcal{L}$ is invariant under the following *global* symmetries:

(i) $\psi \to e^{i\alpha}\psi \Rightarrow \bar{\psi} \to e^{-i\alpha}\bar{\psi} \Rightarrow \mathcal{L}$ is invariant under $U(1)$ global symmetry.

The corresponding current is $j_\mu = \bar{\psi}\gamma^\mu\psi$.

It is conserved: $\partial_\mu j^\mu = 0$

as $\gamma_5^+ = \gamma_5$

(ii) $\psi \to e^{i\gamma_5\alpha}\psi \Rightarrow \bar{\psi} = \psi^+\gamma^0 \to \psi^+ e^{-i\gamma_5\alpha}\gamma^0$

$\{\gamma_5,\gamma^0\}=0$

$= \psi^+\gamma^0 e^{i\gamma_5\alpha} = \bar{\psi}e^{i\alpha\gamma_5} \Rightarrow$

$$\bar{\psi}\, i\gamma^\mu D_\mu \psi \to \bar{\psi}e^{i\alpha\gamma_5}\, i\gamma^\mu D_\mu e^{i\alpha\gamma_5}\psi =$$

$$= \bar{\psi}e^{i\alpha\gamma_5}e^{-i\alpha\gamma_5}\, i\gamma^\mu D_\mu\psi = \bar{\psi}\, i\gamma^\mu D_\mu\psi$$

as $\{\gamma_5,\gamma^\mu\}=0$

$\Rightarrow$ corresponding conserved current is

$j_\mu^5 = \bar{\psi}\gamma^\mu\gamma_5\psi$: $\partial_\mu j^{5\mu} = 0$

$\Rightarrow$ seems like massless QED lagrangian is invariant under the axial symmetry $U_A(1)$

$\Rightarrow \mathcal{L}_{QED}$ is $U(1) \otimes U_A(1)$ invariant.

However, this is <u>not true</u> when quantum corrections are included. $\Rightarrow$ we will see that $\partial_\mu j^{5\mu} \neq 0$ if quantum corrections are counted.

Consider $\partial_\mu j^{5\mu} = \partial_\mu (\bar{\psi}\gamma^\mu\gamma_5\psi) \Rightarrow$ in momentum space $\partial_\mu \to -ik_\mu$, the vertex has $\gamma^\mu\gamma_5$.

Consider 3-point correlator:

$$T_{\mu\nu\rho}(k_1,k_2) = i\int d^4x_1\, d^4x_2\, e^{ik_1\cdot x_1 + ik_2\cdot x_2} \langle 0 | T\big(j_\mu(x_1) \cdot j_\nu(x_2) \cdot j^5_\rho(0)\big)|0\rangle$$

~~Writing $T_{\mu\nu\rho}(k_1,k_2) = i\int d^4x_1\, d^4x_2\, e^{i(x_1-y)k_1 + ik_2(x_2-y)}$ $\langle 0|T[j_\mu(x_1) j_\nu(x_2) j^5_\rho(y)]|0\rangle$ $\Rightarrow$ we expect $(k_1+k_2)^\rho T_{\mu\nu\rho}(k_1,k_2) = i\int d^4x_1\, d^4x_2\; i\,\partial^\rho_y[e^{ik_1(x_1-y)+ik_2(x_2-y)}]$ $\langle 0|T[j_\mu(x_1) j_\nu(x_2) j^5_\rho(y)]|0\rangle$ $\Rightarrow$ (parts)~~

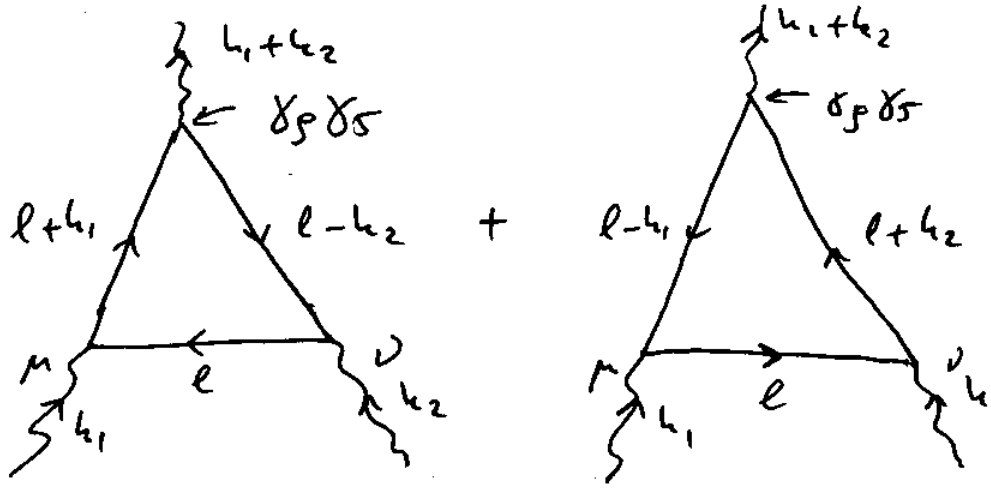
One can show that $\partial_\mu j^{5\mu} = 0$ would lead to

$$(k_1+k_2)^\rho\, T_{\mu\nu\rho}(k_1,k_2) = 0 .$$

Check this statement:

$T_{\mu\nu\rho}(k_1, k_2) =$

arrow indicates both momentum & fermion #.



graph A   graph B

(Can write $T_{\mu\nu\rho}(k_1,k_2) \underset{\substack{x_1\to 0\\ 0\to x_1}}{=} i\int d^4x_1\, d^4x_2\, e^{-ik_1\cdot x_1 + ik_2\cdot(x_2-x_1)}$

$\cdot\langle 0|T\left[j_\mu(0)\, j_\nu(x_2)\, j_\rho^5(x_1)\right]|0\rangle \Rightarrow (k_1+k_2)^\rho\, T_{\mu\nu\rho} =$

$= i\int d^4x_1\, d^4x_2\; i\,\partial^\rho_{x_1}\left(e^{ik_2\cdot x_2 - i(k_1+k_2)\cdot x_1}\right)\langle 0|T\left[j_\mu(0)j_\nu(x_2)j_\rho^5(x_1)\right]|0\rangle$

$= (\text{parts}) = \int d^4x_1\, d^4x_2\, e^{ik_2\cdot x_2 - i(k_1+k_2)\cdot x_1}\langle 0|T\left[j_\mu(0)j_\nu(x_2)\cdot\right.$

$\left.\cdot\,\partial^\rho j_\rho^5(x_1)\right]|0\rangle = 0$ if $\partial^\rho j_\rho^5 = 0$.)

fermion loop

$$-i\,T_{\mu\nu\rho} = -(-ie)^2\int\frac{d^4\ell}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma_\rho\gamma_5\frac{i}{\not\ell+\not k_1}\gamma_\mu\frac{i}{\not\ell}\gamma_\nu\frac{i}{\not\ell-\not k_2}\right]$$

$$-(-ie)^2\int\frac{d^4\ell}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma_\rho\gamma_5\frac{i}{\not\ell+\not k_2}\gamma_\nu\frac{i}{\not\ell}\gamma_\mu\frac{i}{\not\ell-\not k_1}\right] =$$

$$= -ie^2\int\frac{d^4\ell}{(2\pi)^4}\left\{\frac{\mathrm{Tr}\left[\gamma_\rho\gamma_5(\not\ell+\not k_1)\gamma_\mu\not\ell\gamma_\nu(\not\ell-\not k_2)\right]}{(\ell^2+i\varepsilon)((\ell+k_1)^2+i\varepsilon)((\ell-k_2)^2+i\varepsilon)}\right. -$$

$$-ie^2\int\frac{d^4\ell}{(2\pi)^4}\ \frac{\mathrm{Tr}[\gamma_\rho\gamma_5(\not\ell+\not k_2)\gamma_\nu\not\ell\gamma_\mu(\not\ell-\not k_1)]}{(\ell^2+i\varepsilon)((\ell+k_2)^2+i\varepsilon)((\ell-k_1)^2+i\varepsilon)}$$

$$\Rightarrow (k_1+k_2)^\rho\, T_{\mu\nu\rho} = e^2\int\frac{d^4\ell}{(2\pi)^4}\left\{\underbrace{\frac{\mathrm{Tr}[(\not k_1+\not k_2)\gamma_5(\not\ell+\not k_1)\gamma_\mu\not\ell\gamma_\nu(\not\ell-\not k_2)]}{(\ell^2+i\varepsilon)((\ell+k_1)^2+i\varepsilon)((\ell-k_2)^2+i\varepsilon)}}_{=A}\right.$$

$$\left.+\underbrace{\frac{\mathrm{Tr}[(\not k_1+\not k_2)\gamma_5(\not\ell+\not k_2)\gamma_\nu\not\ell\gamma_\mu(\not\ell-\not k_1)]}{(\ell^2+i\varepsilon)((\ell+k_2)^2+i\varepsilon)((\ell-k_1)^2+i\varepsilon)}}_{=B}\right\}$$

Numerator of A $= \mathrm{Tr}\Big[(\not k_1+\not\ell-(\not\ell-\not k_2))\gamma_5(\not\ell+\not k_1)\gamma_\mu\not\ell$
$\gamma_\nu(\not\ell-\not k_2)\Big] = -(\ell+k_1)^2\,\mathrm{Tr}[\gamma_5\gamma_\mu\not\ell\gamma_\nu(\not\ell-\not k_2)] -$
$-(\ell-k_2)^2\,\mathrm{Tr}[\gamma_5(\not\ell+\not k_1)\gamma_\mu\not\ell\gamma_\nu]$

Numerator of B $= \mathrm{Tr}\Big[((\not k_2+\not\ell)-(\not\ell-\not k_1))\gamma_5(\not\ell+\not k_2)\cdot$
$\cdot\gamma_\nu\not\ell\gamma_\mu(\not\ell-\not k_1)\Big] = -(\ell+k_2)^2\,\mathrm{Tr}[\gamma_5\gamma_\nu\not\ell\gamma_\mu(\not\ell-\not k_1)]$
$-(\ell-k_1)^2\,\mathrm{Tr}[\gamma_5(\not\ell+\not k_2)\gamma_\nu\not\ell\gamma_\mu] \Rightarrow$

$$(k_1+k_2)^\rho T_{\mu\nu\rho} = -e^2\int\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{\ell^2+i\varepsilon}\left\{\frac{\mathrm{Tr}[\gamma_5\gamma_\mu\not\ell\gamma_\nu(\not\ell-\not k_2)]}{(\ell-k_2)^2+i\varepsilon}^{①}+\right.$$

$$+\frac{\mathrm{Tr}[\gamma_5(\not\ell+\not k_1)\gamma_\mu\not\ell\gamma_\nu]}{(\ell+k_1)^2+i\varepsilon}^{②}+\frac{\mathrm{Tr}[\gamma_5\gamma_\nu\not\ell\gamma_\mu(\not\ell-\not k_1)]}{(\ell-k_1)^2+i\varepsilon}^{③}+$$

$$\left.+\frac{\mathrm{Tr}[\gamma_5(\not\ell+\not k_2)\gamma_\nu\not\ell\gamma_\mu]}{(\ell+k_2)^2+i\varepsilon}^{④}\right\}$$

Now, if in 4 we shift $\ell \to \ell - k_2 \Rightarrow$ it will cancel 1 as $1 + 4 \propto \{\gamma_5, \gamma_\mu\} = 0$.

In 3 shift $\ell \to \ell + k_1 \Rightarrow$ cancel 2.

$\Rightarrow$ seems to get $(k_1 + k_2)^\rho T_{\mu\nu\rho} = 0$ in expectation with $\partial^\rho j^5_\rho = 0$ ...

Problem at large-ℓ all integrals are quadratically divergent!

We get $1 \sim 2 \sim 3 \sim 4 \sim \int d^4\ell \frac{1}{\ell^2} \sim \int^\infty d\ell \cdot \ell \sim \infty^2$.

$\Rightarrow$ can't shift variables in divergent integrals!

$$\int_0^\infty d\ell \cdot \ell \xrightarrow[\ell \to \ell + a]{\text{shift}} \int_{-a}^\infty d\ell \cdot (\ell + a) = \int_0^\infty d\ell \cdot (\ell + a) + \int_{-a}^0 d\ell \cdot (\ell + a)$$

$$= \int_0^\infty d\ell \cdot \ell + a \cdot \int_0^\infty d\ell + \left(\frac{\ell^2}{2} + a\ell\right)\Big|_{-a}^0 = \underbrace{\int_0^\infty d\ell \cdot \ell}_{\text{old integral}} + a \underbrace{\int_0^\infty d\ell}_{\infty} + \frac{a^2}{2}$$

$\Rightarrow$ did not survive the shift, got corrections?

$\Rightarrow$ ill-defined procedure $\Rightarrow$ need to make integrals finite, need to regulate them!

We'll use Pauli-Villars regularization: introduce (subtract) a new particle with mass m, which is then taken to ∞ to eliminate the particle.