Last time: started to talk about

axial Anomaly (cont'd)

massless QED: $U(1) \otimes U(1)_A$ symmetric at the classical Lagrangian level.

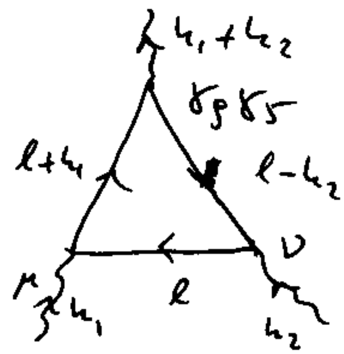
To check if it's true at quantum level defined the correlator:

$$T_{\mu\nu\rho}(k_1,k_2) = i\int d^4x_1\, d^4x_2\, e^{ik_1\cdot x_1 + ik_2\cdot x_2} \langle 0|\, T\big(j_\mu(x_1)\, j_\nu(x_2) \cdot j^5_\rho(0)\big)\, |0\rangle$$

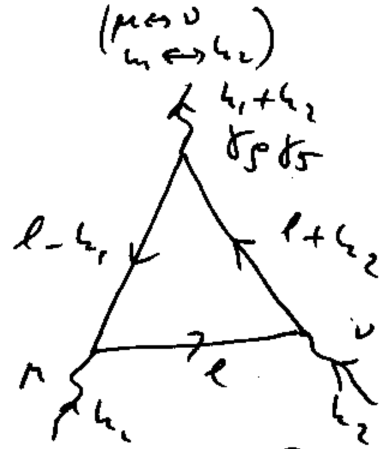
* showed that if $\partial_\rho j^{5\rho} = 0 \Rightarrow \boxed{(k_1+k_2)_\rho T^{\mu\nu\rho} = 0}$.

Diagrammatically

$-i\,T_{\mu\nu\rho}(k_1,k_2) =$



$+$



$$\Rightarrow \text{got } (k_1+k_2)^\rho T_{\mu\nu\rho} = -e^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2+i\varepsilon} \Bigg\{ \overset{①}{\frac{\mathrm{Tr}[\gamma_5\gamma_\mu \not{\ell}\gamma_\nu(\not{\ell}-\not{k}_2)]}{(\ell-k_2)^2+i\varepsilon}}$$

$$+ \overset{②}{\frac{\mathrm{Tr}[\gamma_5(\not{\ell}+\not{k}_1)\gamma_\mu\not{\ell}\gamma_\nu]}{(\ell+k_1)^2+i\varepsilon}} + \overset{③}{\frac{\mathrm{Tr}[\gamma_5\gamma_\nu\not{\ell}\gamma_\mu(\not{\ell}-\not{k}_1)]}{(\ell-k_1)^2+i\varepsilon}} + \overset{④}{\frac{\mathrm{Tr}[\gamma_5(\not{\ell}+\not{k}_2)\gamma_\nu\not{\ell}\gamma_\mu]}{(\ell+k_2)^2+i\varepsilon}} \Bigg\}$$

Shifting $\ell \to \ell - k_2$ in ④ got ① + ④ = 0

—"— $\ell \to \ell + k_1$ in ③ got ② + ③ = 0.

Is $(k_1 + k_2)^\rho \, T_{\mu\nu\rho} = 0$? Not clear yet, as the integrals over ℓ in all 4 terms are infinite!

Now, if in 4 we shift $\ell \to \ell - k_2$ => it will cancel 1 as $1 + 4 \propto \{\gamma_5, \gamma_\mu\} = 0$.

In 3 shift $\ell \to \ell + k_1$ => cancel 2.

=> seems to get $(k_1+k_2)^\rho T_{\mu\nu\rho} = 0$ in expectation with $\partial^\rho j^5_\rho = 0$ ...

<u>Problem</u> at large-ℓ all integrals are quadratically divergent!

We get $1 \sim 2 \sim 3 \sim 4 \sim \int d^4\ell \frac{1}{\ell^2} \sim \int^\infty d\ell \cdot \ell \sim \infty^2$.

=> can't shift variables in divergent integrals!

$$\int_0^\infty d\ell \cdot \ell \xrightarrow[\substack{\text{shift} \\ \ell \to \ell + a}]{} \int_{-a}^\infty d\ell \cdot (\ell + a) = \int_0^\infty d\ell \cdot (\ell + a) + \int_{-a}^0 d\ell \cdot (\ell + a)$$

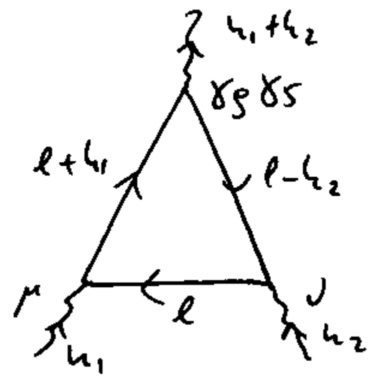
$$= \int_0^\infty d\ell \cdot \ell + a \cdot \int_0^\infty d\ell + \left(\frac{\ell^2}{2} + a\ell\right)\Big|_{-a}^0 = \underbrace{\int_0^\infty d\ell \cdot \ell}_{\text{old integral}} + a \underbrace{\int_0^\infty d\ell}_{\infty} + \frac{a^2}{2}$$

=> did not survive the shift, got corrections?

=> ill-defined procedure => need to make integrals finite, need to regulate them!

We'll use Pauli-Villars regularization: introduce (subtract) a new particle with mass m, which is then taken to ∞ to eliminate the particle.

$T_{\mu\nu\rho}$ =



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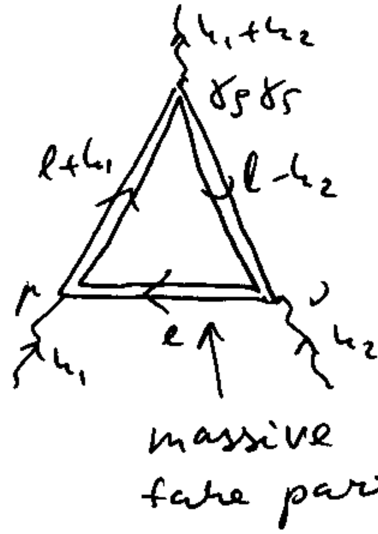


$+ \underbrace{\begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}}_{\text{graph B}}$

$$T_{\mu\nu\rho} = -e^2 \int \frac{d^4 l}{(2\pi)^4} \left\{ \frac{Tr[\gamma_\rho \gamma_5 (\not{l}+\not{k}_1)\gamma_\mu \not{l} \gamma_\nu (\not{l}-\not{k}_2)]}{(l^2+i\varepsilon)((l+k_1)^2+i\varepsilon)((l-k_2)^2+i\varepsilon)} \right.$$

$$\left. - \frac{Tr[\gamma_\rho\gamma_5(\not{l}+\not{k}_1+m)\gamma_\mu(\not{l}+m)\gamma_\nu(\not{l}-\not{k}_2+m)]}{(l^2-m^2+i\varepsilon)((l+k_1)^2-m^2+i\varepsilon)((l-k_2)^2-m^2+i\varepsilon)} \right\} + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}$$

The second Tr has only even powers of m in its expansion: m^0 & m^2. (Tr of odd # of γ's is zero.) Write:

$$(k_1+k_2)^\rho \, T_{\mu\nu\rho} = -e^2 \int \frac{d^4 l}{(2\pi)^4} \left\{ Tr[(\not{k}_1+\not{k}_2)\gamma_5(\not{l}+\not{k}_1)\gamma_\mu \not{l} \gamma_\nu \cdot \right.$$

$$\cdot (\not{l}-\not{k}_2)] \left[\frac{1}{l^2(l+k_1)^2(l-k_2)^2} - \frac{1}{(l^2-m^2)[(l+k_1)^2-m^2][(l-k_2)^2-m^2]} \right] -$$

$$\left. - \frac{m^2\text{-term in 2nd trace } (O(l))}{[l^2-m^2][(l+k_1)^2-m^2][(l-k_2)^2-m^2]} \right\} + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}$$

Now the integral is convergent & shifts are allowed!

$\Rightarrow$ the $m=0$ (first) term in [...] vanishes like before.

For the term in [...] containing m^2 write:

$$\mathrm{Tr}\left[\left(\not k_1+\not l-(\not l-\not k_2)\right)\gamma_5(\not l+\not k_1)\gamma_\mu \not l \gamma_\nu(\not l-\not k_2)\right] =$$

$$= -(l+k_1)^2\,\mathrm{Tr}\left[\gamma_5\gamma_\mu \not l\gamma_\nu(\not l-\not k_2)\right]-(l-k_2)^2\,\mathrm{Tr}\left[\gamma_5(\not l+\not k_1)\cdot\right.$$

$$\left.\cdot\gamma_\mu\not l\gamma_\nu\right] = -\left[(l+k_1)^2-m^2\right]\,\mathrm{Tr}\left[\gamma_5\gamma_\mu\not l\gamma_\nu(\not l-\not k_2)\right]$$

$$-\left[(l-k_2)^2-m^2\right]\mathrm{Tr}\left[\gamma_5(\not l+\not k_1)\gamma_\mu\not l\gamma_\nu\right]-m^2\Big(\mathrm{Tr}\left[\gamma_5\gamma_\mu\not l\gamma_\nu\cdot\right.$$

$$\left.\cdot(\not l-\not k_2)\right]+\mathrm{Tr}\left[\gamma_5(\not l+\not k_1)\gamma_\mu\not l\gamma_\nu\right]\Big)$$

First two terms also cancel after shifts.

We get:

$$(k_1+k_2)^\rho\,T_{\mu\nu\rho} = -e^2\int\frac{d^4l}{(2\pi)^4}\,m^2\left\{\frac{1}{\left[l^2-m^2\right]\left[(l+k_1)^2-m^2\right]\left[(l-k_2)^2-m^2\right]}\right.$$

$$\cdot\Big\{\mathrm{Tr}\left[\gamma_5\gamma_\mu\not l\gamma_\nu(\not l-\not k_2)\right]+\mathrm{Tr}\left[\gamma_5(\not l+\not k_1)\gamma_\mu\not l\gamma_\nu\right]-$$

$$-\mathrm{Tr}\left[(\not k_1+\not k_2)\gamma_5\gamma_\mu\gamma_\nu(\not l-\not k_2)\right]-\mathrm{Tr}\left[(\not k_1+\not k_2)\gamma_5\cdot\right.$$

$$\left.\cdot(\not l+\not k_1)\gamma_\mu\gamma_\nu\right]-\mathrm{Tr}\left[(\not k_1+\not k_2)\gamma_5\gamma_\mu\not l\gamma_\nu\right]\Big\}+\begin{pmatrix}k_1\leftrightarrow k_2\\ \mu\leftrightarrow\nu\end{pmatrix}\Bigg\}\begin{matrix}m^2\\ \text{terms}\\ \text{in}\\ \mathrm{Tr.}\end{matrix}$$

$$=\left(\text{as }\mathrm{Tr}\left[\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right]=-4i\,\varepsilon^{\mu\nu\alpha\beta}\right) =$$

$$= 4 i e^2 \varepsilon^{\mu\nu\alpha\beta} \int \frac{d^4 l}{(2\pi)^4} \frac{m^2}{(l^2-m^2)[(l+k_1)^2-m^2][(l-k_2)^2-m^2]}$$

$$\cdot \Big\{ -l_\alpha (\cancel{l}-k_2)_\beta + l_\alpha(\cancel{l}+k_1)_\beta - (l-k_2)_\alpha (k_1+k_2)_\beta + (k_1+k_2)_\alpha (l+k_1)_\beta$$

$$- (k_1+k_2)_\alpha l_\beta \Big\} + \begin{pmatrix} k_1\leftrightarrow k_2 \\ \mu\leftrightarrow\nu \end{pmatrix} = 4ie^2 \varepsilon^{\mu\nu\alpha\beta}\int\frac{d^4l}{(2\pi)^4} \cdot$$

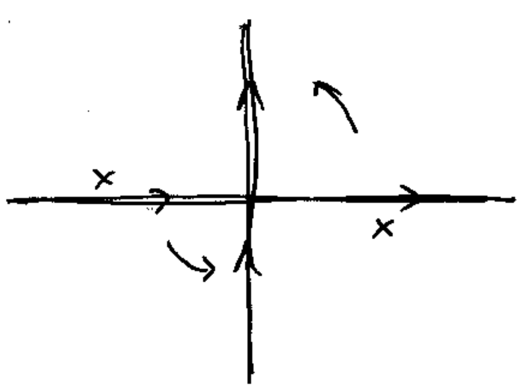
$$\cdot \frac{m^2}{[l^2-m^2][(l+k_1)^2-m^2][(l-k_2)^2-m^2]} \Big\{ \cancel{l_\alpha k_{2\beta} + l_\alpha k_{1\beta}} -$$

$$- \cancel{l_\alpha(k_1+k_2)_\beta} + \cancel{(k_1+k_2)_\alpha l_\beta} - \cancel{(k_1+k_2)_\alpha l_\beta} + k_{2\alpha}(k_1+\underset{0}{\cancel{k_2}})_\beta$$

$$+ k_{1\beta}(\underset{0}{\cancel{k_1}}+k_2)_\alpha \Big\} + \begin{pmatrix} \mu\leftrightarrow\nu \\ k_1\leftrightarrow k_2 \end{pmatrix} = 8 i e^2 \varepsilon^{\mu\nu\alpha\beta} k_{1\beta} k_{2\alpha} \cdot \underbrace{\int \frac{d^4 l}{(2\pi)^4} \cdot \frac{m^2}{(l^2-m^2)[(l+k_1)^2-m^2][(l-k_2)^2-m^2]}}_{I_m} + \begin{pmatrix} k_1\leftrightarrow k_2 \\ \mu\leftrightarrow\nu \end{pmatrix}$$

Approximate the integral by: ($l, m \sim$ large)

$$I_m \approx \int \frac{d^4 l}{(2\pi)^4} \frac{m^2}{[l^2-m^2+i\varepsilon]^3} = \Bigg| \begin{array}{l} \text{Wick rotation} \\ l_0 = + i\, l_0^E \end{array}$$



$$l^2 - m^2 + i\varepsilon = \left(l_0 - \sqrt{\vec{l}^2+m^2} + i\varepsilon\right) \cdot \left(l_0 + \sqrt{\vec{l}^2+m^2} - i\varepsilon\right)$$

$$\Rightarrow I_m = -i\int \frac{d^4 l_E}{(2\pi)^4}\,\frac{m^2}{\left[l_E^2+m^2\right]^3} = -i\int_0^\infty \frac{l_E^3\, dl_E}{(2\pi)^4}\cdot \underbrace{\int d\Omega_4}_{2\pi^2}\cdot$$

$$\cdot\frac{m^2}{\left[l_E^2+m^2\right]^3} = -i\,\frac{1}{8\pi^2}\,m^2\int_0^\infty \frac{dl\cdot l^3}{\left[l^2+m^2\right]^3} = -i\,\frac{1}{16\pi^2}\,m^2\cdot$$

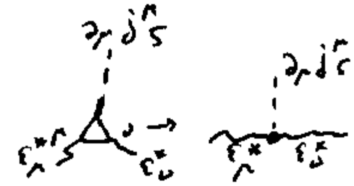
$$\cdot\int_0^\infty \frac{dl^2\cdot\left[l^2+m^2-m^2\right]}{\left[l^2+m^2\right]^3} = -i\,\frac{1}{(4\pi)^2}\,m^2\cdot\left[\frac{1}{m^2}-m^2\frac{1}{2m^4}\right] =$$

$$= -i\,\frac{1}{2}\,\frac{1}{(4\pi)^2}.$$ We get

$$\begin{pmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{pmatrix}\downarrow$$

$$(k_1+k_2)^\rho\, T_{\mu\nu\rho} = 8\cancel{i}e^2\,\varepsilon^{\mu\nu\alpha\beta}\,k_{1\beta}\,k_{2\alpha}\,\cancel{(-i)}\,\frac{1}{2}\,\frac{1}{(4\pi)^2}\,2 \Rightarrow$$

$$(k_1+k_2)^\rho\, T_{\mu\nu\rho} = -2\frac{\alpha_{EM}}{\pi}\,\varepsilon^{\mu\nu\alpha\beta}\,k_{1\alpha}\,k_{2\beta}$$



$\Rightarrow$ in operator language this means:

$$\boxed{\partial_\mu j^\mu_5 = \frac{-\alpha}{4\pi}\,\varepsilon^{\mu\nu\alpha\beta}\,F_{\mu\nu}F_{\alpha\beta}}$$ Adler-Bell-Jackiw anomaly '69

$\Rightarrow$ classically conserved current is <u>not</u> conserved quantum mechanically!

$\Rightarrow$ in QED this ABJ anomaly relation is exact $\sim$ no higher-order corrections.

In QCD have $j_{5\mu} = \sum_f \bar{q}_f \gamma_\mu \gamma_5 q_f$

and $$\boxed{\partial_\mu j_5^\mu = -\frac{\alpha_s N_f}{8\pi} \varepsilon^{\mu\nu\alpha\beta} F^a_{\mu\nu} F^a_{\rho\sigma}}.$$

$\Rightarrow$ $U(1)_A$ in QCD is broken, but has no Goldstone boson associated with this breaking $\Rightarrow$ symmetry was never there in the full quantum theory

(Otherwise, if treating $U(1)_A$ as a symmetry, would expect parity-doubling of baryon states. If $U(1)_A$ is broken $\sim$ expect Goldstone modes. This way we see that the symmetry is never a good symmetry.)

$\Rightarrow$ to get $\neq 0$ $\varepsilon^{\mu\nu\alpha\beta} F^a_{\mu\nu} F^a_{\rho\sigma}$ need instantons ... (later)

$\Rightarrow$ axial anomaly is responsible for pion decay : $\pi^0 \to \gamma\gamma$