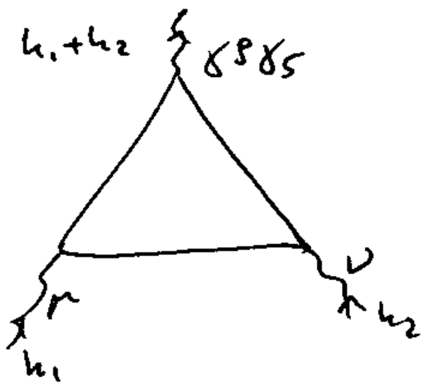
Last time: finished calculating the triangle diagram & got



$$(k_1+k_2)^\rho\, T_{\mu\nu\rho}(k_1,k_2) = -2\,\frac{\alpha_{EM}}{\pi}\cdot \varepsilon^{\mu\nu\alpha\beta}\, k_{1\alpha}\, k_{2\beta}$$

In the operator terms this means

$$\boxed{\partial_\mu j_5^\mu = -\frac{\alpha_{EM}}{4\pi}\,\varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}}$$

$\Rightarrow$ axial current is not conserved at the quantum level

$\Rightarrow$ $U(1)_A$ is <u>not</u> (and never was) a symmetry of the full quantum theory

$\Rightarrow$ if a symmetry is there classically, but is broken at the quantum level ~ <u>anomaly</u>.

In QCD: $$\boxed{\partial_\mu j_5^\mu = -\frac{\alpha_s N_f}{8\pi}\,\varepsilon^{\mu\nu\alpha\beta} F^a_{\mu\nu} F^a_{\alpha\beta}}$$

As the theory is never there $\Rightarrow$ no Goldstone bosons $\Rightarrow$ no light isosinglet pseudoscalar meson with mass $\approx m_\pi$

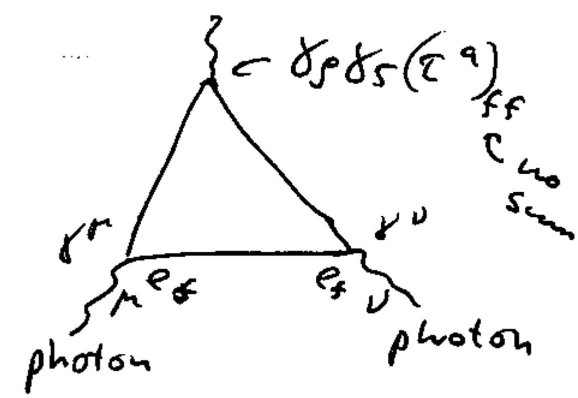
(η-meson ~ a candidate, but too heavy)

$\pi^0 \to \gamma\gamma$

Consider axial isospin current $j^a_{5\mu} = \bar{q}\gamma_\mu\gamma_5\tau^a q$

where τ^a = Pauli matrices, $a = 1, 2, 3$ (flavor index for $SU(2)$ flavor). Here $q = \begin{pmatrix} u \\ d \end{pmatrix}$.

It has an anomaly due to quarks coupling to photons:



$$\partial_\mu j_5^{a\mu} = -\frac{\alpha_{EM}}{4\pi}\varepsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu} \cdot \sum_f (\tau^a)_{ff} \cdot e_f^2$$

as $\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\Rightarrow$ only τ^3 gives $\neq 0$ anomaly

$$\sum_f (\tau^3)_{ff}\, e_f^2 = \left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2 = \frac{1}{3}$$

$$\Rightarrow \partial_\mu j_5^{3\mu} = -\frac{\alpha_{EM}}{12\pi}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$$

$j_5^{3\mu} = \bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma_\mu\gamma_5 d$ annihilates π^0:

$$\langle 0 | j_5^{3\mu}(0) | \pi^0(p)\rangle = i f_\pi p^\mu$$

(due to spont. chiral symm. breaking)

with $f_\pi \simeq 93\ MeV$ (pion decay constant)

$\Rightarrow$ in general $\langle 0 | j^{3\mu}_5(x) | \pi^0(p) \rangle = i p^\mu f_\pi e^{-ipx}$

$\Rightarrow \langle 0 | \partial_\mu j^{3\mu}_5(x) | \pi^0(p) \rangle = \underbrace{p_\mu p^\mu}_{m_\pi^2} f_\pi e^{-ip\cdot x}$

$\Rightarrow \langle 0 | \partial_\mu j^{3\mu}_5(0) | \pi^0(p) \rangle = m_\pi^2 f_\pi$

$\Rightarrow$ pion couples to $\partial_\mu j^{3\mu}_5 \sim \varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F^{\alpha\beta}$

$\Rightarrow \sim A_\rho A_\sigma \Rightarrow$ pion couples to two photons

$\Rightarrow$ can have $\pi^0 \to \gamma\gamma$ decay due to the axial anomaly.

Axial anomaly in the Standard Model.

$\Rightarrow$ a theory with axial anomaly would violate Ward identities $\left((k_1+k_2)^\rho T_{\mu\nu\rho}=0\right)$, an. is therefore not gauge invariant!

$\Rightarrow$ this would be a problem for theories with axial current coupling to gauge bosons $\binom{e.g.}{SM}$

$\Rightarrow$ in particular an anomaly would spoil renormalizability of the theory

$\Rightarrow$ Standard model has vector bosons coupling with γ_5 to leptons and quarks. For SM to be consistent need those 3-boson couplings with γ_5 to cancel!

Let's go back to SM lagrangian:

$$\boxed{\begin{aligned}\mathcal{L} &= \bar{R}_e i\gamma^\mu\left(\partial_\mu + i\frac{g'}{2}YB_\mu\right)R_e + \bar{L}_e i\gamma^\mu\left(\partial_\mu + i\frac{g'}{2}YB_\mu - ig\frac{\vec{\tau}}{2}\cdot\vec{W}_\mu\right)L_e + (\mu,\tau) + \bar{L}_u i\gamma^\mu\left(\partial_\mu + i\frac{g'}{2}YB_\mu - ig\frac{\vec{\tau}}{2}\cdot\vec{W}_\mu\right) \\ &\quad \cdot L_u + \bar{R}_u i\gamma^\mu\left(\partial_\mu + i\frac{g'}{2}YB_\mu\right)R_u + \bar{R}_d i\gamma^\mu\left(\partial_\mu + i\frac{g'}{2}YB_\mu\right)R_d \\ &\quad + (2\text{ more generations}) + \ldots\end{aligned}}$$

(we keep quark/lepton-vector boson terms only)

Y is the weak hyperdcharge

$$\boxed{Q = I_3 + \frac{Y}{2}}$$

Gell-Mann–Nishijima relation always holds.

$\Rightarrow$ for L_e : $I_3 = \pm 1/2$; $Q = 0$ for neutrino

$\Rightarrow$ $0 = \frac{1}{2} + \frac{Y}{2} \Rightarrow$ $Y_{L_e} = -1$

for R_e have $I_3 = 0$, $Q = -1 \Rightarrow -1 = \frac{Y}{2} \Rightarrow$ $Y_{R_e} = -2$

for L_u: u-quark has $Q = +2/3 \Rightarrow \frac{2}{3} = \frac{1}{2} + \frac{Y}{2}$

$\Rightarrow$ $Y_{L_u} = 1/3$

for R_u: $I_3 = 0 \Rightarrow \frac{2}{3} = \frac{Y}{2} \Rightarrow$ $Y_{R_u} = 4/3$

for R_d : $Q = -1/3 \Rightarrow -1/3 = \frac{Y}{2} \Rightarrow$ $Y_{R_d} = -2/3$

other generations r same $\Rightarrow$ forget about them.

As $L_e = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L = \frac{1-\gamma_5}{2} \begin{pmatrix} \nu_e \\ e \end{pmatrix}$, $R_e = \frac{1+\gamma_5}{2} e = e_R$

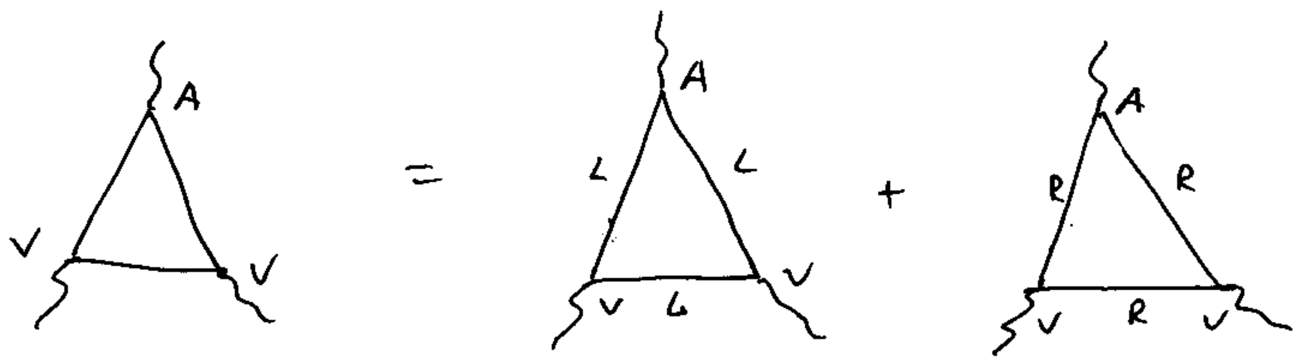
$\Rightarrow$ all W_μ, B_μ couplings involve γ_5 $\Rightarrow$ need divergences to cancel.

massless QED

$$\mathcal{L}_{QED} = \bar{\psi}\, i\gamma^\mu D_\mu \psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \bar{\psi}_L\, i\gamma^\mu D_\mu \psi_L$$

$$+ \bar{\psi}_R\, i\gamma^\mu D_\mu \psi_R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

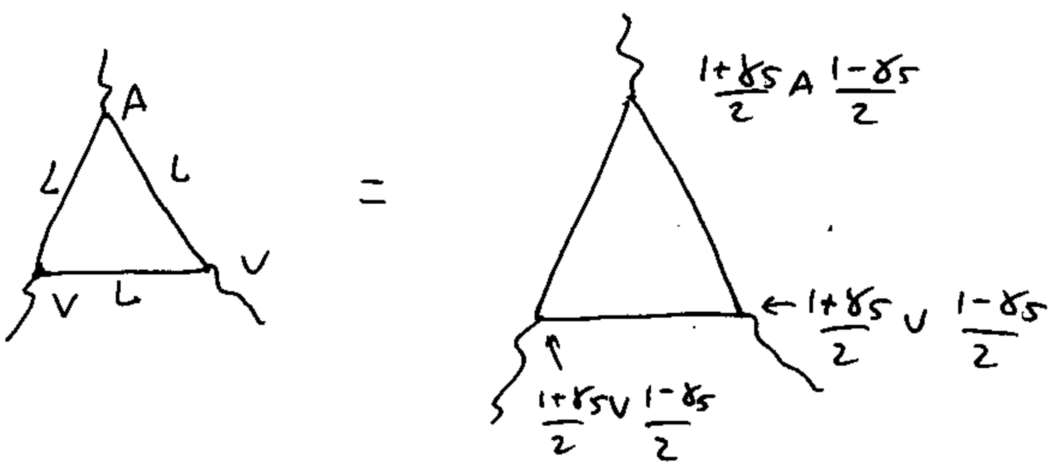
$\Rightarrow$ the anomaly consists of left-handed and right-handed electrons' contributions



$A = \gamma_\rho \gamma_5$

$V = \gamma_\mu$, or γ_ν

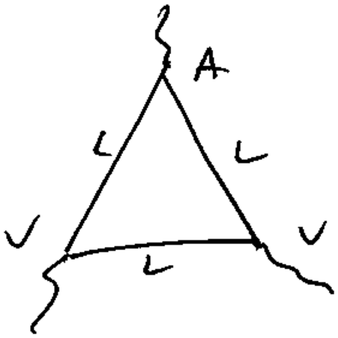
Propagator $\langle \psi_L \bar{\psi}_L \rangle = \langle \frac{1-\gamma_5}{2}\psi\, \bar{\psi}\, \frac{1+\gamma_5}{2} \rangle \Rightarrow$



$$\frac{1+\gamma_5}{2}\gamma_\mu \frac{1-\gamma_5}{2} = \gamma_\mu \frac{1-\gamma_5}{2} = \frac{V-A}{2}$$

$$\frac{1+\gamma_5}{2}\gamma_\rho\gamma_5 \frac{1-\gamma_5}{2} = \gamma_\rho\gamma_5\frac{1-\gamma_5}{2} = \gamma_\rho \frac{\gamma_5 - 1}{2} = \frac{A-V}{2}$$

Hence



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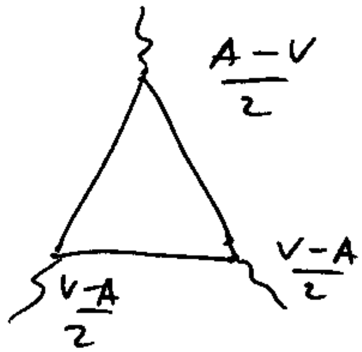
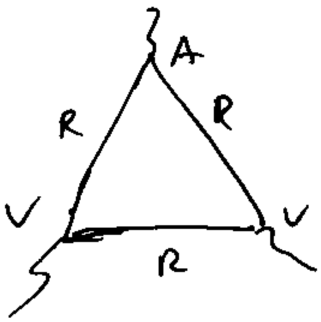


$= \frac{1}{2}$ Anomaly

Similarly



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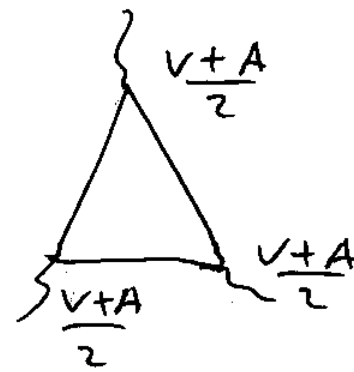


$= \frac{1}{2}$ Anomaly

Subtract, get

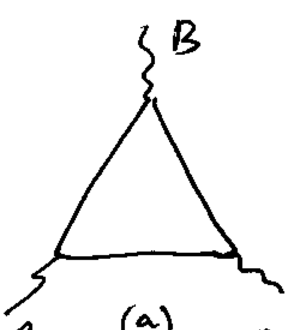
$= 0$

$\Rightarrow$ anomalies cancel!

No anomaly in 3-boson coupling! (in QED)

$\Rightarrow$ in SM need to sum all graphs with left- and right- handed particles in the loop.

The diagrams are:



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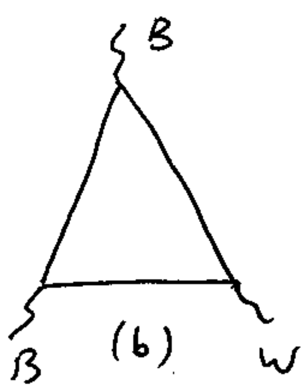

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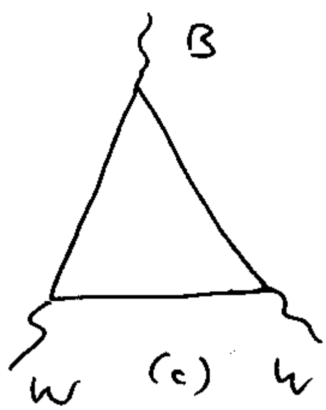
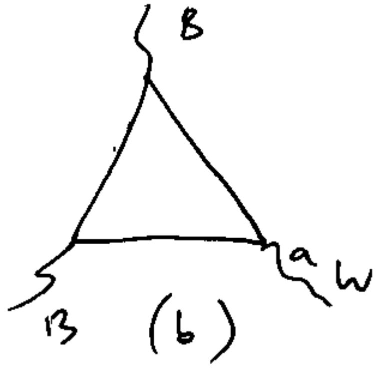
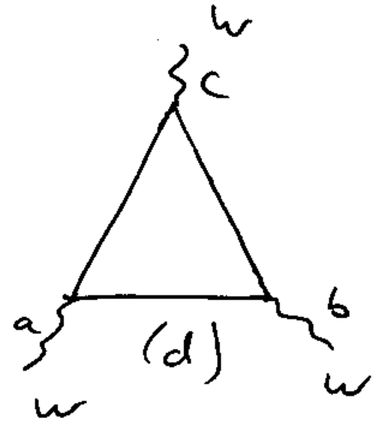


First let's do (b): $\mathrm{tr}\,\tau^a = 0 \Rightarrow \boxed{(b) = 0.}$



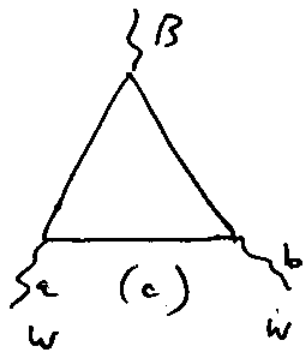
Now, let's look at (d):



$$\mathrm{tr}\left(\tau^c \tau^a \tau^b\right) + \mathrm{tr}\left(\tau^c \tau^b \tau^a\right)$$

$$= \mathrm{tr}\left[\tau^c \underbrace{\{\tau^a, \tau^b\}}_{2\delta^{ab}}\right] \sim \mathrm{tr}\,\tau^c = 0 \Rightarrow \boxed{(d) = 0}$$

Next let's look at (c):



$\mathrm{tr}\,\frac{\tau^a}{2}\frac{\tau^b}{2} = \frac{1}{2}\delta^{ab}$ $\sim$ not zero

$$(c) \propto \sum_{i = \text{left-handed doublets}} Y_i$$ (as W couples to left-handed quarks & leptons only)

$$\Rightarrow (c) \propto Y_{Le} + Y_{Lu} \cdot \underbrace{3}_{\text{No. of colors}} = -1 + \frac{1}{3} \cdot 3 = 0$$

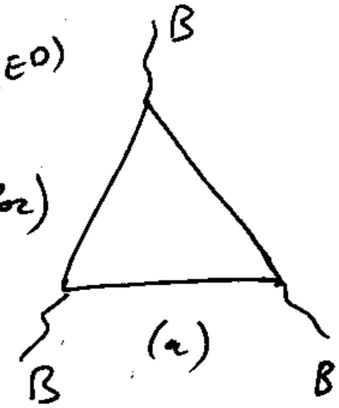
$$\Rightarrow \boxed{(c) = 0}$$

Finally, let's look at (a):

contribute "–" to anomaly (see QED)

$$(a) \propto \underset{\substack{i=\text{left-handed}\\ \text{doublets}}}{2\sum} Y_i^3 \times (\text{color}) - \sum_{\substack{i=\text{right-}\\ \text{-handed}}} Y_i^3 \times (\text{color})$$

doublets



$$= \underset{L_e}{2(-1)^3} + \underset{L_u}{2\cdot\left(\tfrac{1}{3}\right)^3}\cdot \underset{\text{color}}{3} - \underset{R_e}{(-2)^3} - \underset{R_u}{\left(\tfrac{4}{3}\right)^3}\cdot \underset{\text{color}}{3} - \underset{R_d}{\left(-\tfrac{2}{3}\right)^3}\cdot \underset{\text{color}}{3}$$

$$= -2 + \frac{2}{9} + 8 - \frac{64}{9} + \frac{8}{9} = 6 - \frac{54}{9} = 0$$

$$\Rightarrow \quad \boxed{(a) = 0}.$$

$\Rightarrow$ the same applies to the other two generations

$\Rightarrow$ anomalies cancel in 3-vector boson couplings in the SM! Thus Standard model is a consistent (gauge-invariant) and renormalizable theory..: as expected.