Last time: talked about free Dirac field.

$$\boxed{\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi} \Rightarrow \text{EOM is Dirac equation:}$$

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0$$

We found the most general solution of Dirac eqn:

$$\psi(x) = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k} \left\{ \hat{b}_{\vec{k},r}\, u_r(\vec{k}) e^{-ik\cdot x} + \hat{d}^+_{\vec{k},r}\, v_r(\vec{k})\, e^{ik\cdot x} \right\}$$

Found the Hamiltonian: $\boxed{H = \int d^3x \; i\psi^+ \partial_0 \psi}$

Not positive-definite? Plug in ψ:

$$H = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k}\, \varepsilon_k \left[\hat{b}^+_{\vec{k},r}\, \hat{b}_{\vec{k},r} - \hat{d}_{\vec{k},r}\, \hat{d}^+_{\vec{k},r} \right]$$

Still not ≥ 0. Have to use anti-commutators!

$$\left\{ \hat{b}_{\vec{k},r},\, \hat{b}^+_{\vec{k}',r'} \right\} = \left\{ \hat{d}_{\vec{k},r},\, \hat{d}^+_{\vec{k},r} \right\} = (2\pi)^3\, 2\varepsilon_k\, \delta_{rr'}\, \delta(\vec{k} - \vec{k}')$$

with all other $\{,\}$'s zero. Dropping an ∞ get

$$H = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k}\, \varepsilon_k \left[\hat{b}^+_{\vec{k},r}\, \hat{b}_{\vec{k},r} + \hat{d}^+_{\vec{k},r}\, \hat{d}_{\vec{k},r} \right]$$ positive-definite!

office hours → MW 2pm?

Define anti-commutation relations:

$$\boxed{\begin{aligned}&\{\hat{b}_{\vec{k},r}, \hat{b}^+_{\vec{k}',r'}\} = \{\hat{d}_{\vec{k},r}, \hat{d}_{\vec{k}',r'}\} = (2\pi)^3\, 2\mathcal{E}_k\, \delta_{rr'}\, \delta^3(\vec{k}-\vec{k}')\\ &\{\hat{b}_{\vec{k},r}, \hat{b}_{\vec{k}',r'}\} = \{\hat{b}^+_{\vec{k},r}, \hat{b}^+_{\vec{k}',r'}\} = 0\\ &\{\hat{d}_{\vec{k},r}, \hat{d}_{\vec{k}',r'}\} = \{\hat{d}^+_{\vec{k},r}, \hat{d}^+_{\vec{k}',r'}\} = 0\end{aligned}}$$

where $\{\hat{A},\hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ ~ anti-commutator

=> dropping ∞ number get

$$\boxed{H = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3\, 2\mathcal{E}_k}\, \mathcal{E}_k \left[\hat{b}^+_{\vec{k},r}\hat{b}_{\vec{k},r} + \hat{d}^+_{\vec{k},r} d_{\vec{k},r} \right]}$$

Now it's positive-definite!

For the fields get $\{\psi_\alpha(\vec{x},t), \underbrace{\pi_\beta(\vec{x}',t)}_{= i\psi_\beta^+}\} = i\delta_{\alpha\beta}\delta(\vec{x}-\vec{y})$

$$\{\psi_\alpha, \psi_\beta\} = \{\psi^+_\alpha, \psi^+_\beta\} = 0$$

=> all operators anti-commute.

Time evolution: $\left.\begin{aligned} +i\frac{\partial}{\partial t}\psi(x) &= [\psi, H]\\ i\frac{\partial}{\partial t}\bar{\psi}(x) &= [\bar{\psi}, H]\end{aligned}\right\}$ Still uses commutators (can show)

Useful formulas: $\bar{u}_r(\vec{k})\, u_r(\vec{k}) = 2m\,\delta_{rs}$

$$\bar{v}_r(\vec{k})\; v_s(\vec{k}) = -2m\,\delta_{rs}$$

$$u_r^+(\vec{k})\, u_s(\vec{k}) = 2\,\mathcal{E}_k\,\delta_{rs}$$

$$v_r^+(\vec{k})\; v_s(\vec{k}) = 2\,\mathcal{E}_k\,\delta_{rs}$$

$$\sum_{r=1}^{2} u_{r,\alpha}(\vec{k})\; \bar{u}_{r,\beta}(\vec{k}) = \left(\gamma\cdot p + m\right)_{\alpha\beta}$$

$$\sum_{r=1}^{2} v_r(\vec{k})\; \bar{v}_r(\vec{k}) = \gamma\cdot p - m.$$

Gauge Fields (photons)

$A^\mu = (\Phi, \vec{A}) \sim$ 4-vector $\Rightarrow$ one can build a gauge invariant tensor: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (field strength tensor) $\Rightarrow$

the Lagrangian is $\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2).$

EOM: $$\frac{\delta \mathcal{L}}{\delta A_\mu} - \partial_\nu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\nu A_\mu)}\right) = 0$$

$$\frac{\delta \mathcal{L}}{\delta A_\mu} = 0\,, \quad \frac{\delta \mathcal{L}}{\delta(\partial_\nu A_\mu)} = -\frac{1}{4}\frac{\delta(F_{\alpha\beta}F^{\alpha\beta})}{\delta(\partial_\nu A_\mu)} = -\frac{1}{4}\frac{\delta\left((\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial_\rho A_\sigma - \partial_\sigma A_\rho)\, g^{\alpha\sigma} g^{\beta\rho}\right)}{\partial(\partial_\nu A_\mu)} = F^{\mu\nu}$$

$\Rightarrow$ get $\boxed{\partial_\nu F^{\mu\nu} = 0}$ (Maxwell eqn's in vacuum)

One can introduce source current j_μ $\Rightarrow$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu \quad \Rightarrow \text{now } \frac{\delta \mathcal{L}}{\delta A_\mu} = -j^\mu$$

$\Rightarrow$ get $\boxed{\partial_\nu F^{\mu\nu} = -j^\mu}$ (full Maxwell's eqn's)

$$\partial_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu) = -j^\mu$$

$$-\partial^\mu \partial_\nu A^\nu + \Box A^\mu = j^\mu$$

Gauge transformation: $A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda$

$\Rightarrow F_{\mu\nu} \to F'_{\mu\nu} = F_{\mu\nu} \Rightarrow F_{\mu\nu} F^{\mu\nu}$ is gauge-invariant.

What about $j_\mu A^\mu$?

$$j_\mu A^\mu \to j_\mu A'^\mu = j_\mu A^\mu + j_\mu \partial^\mu \Lambda = j_\mu A^\mu + \underbrace{\partial^\mu (j_\mu \Lambda)}_{\substack{\text{surface term} \\ \Rightarrow \text{discard}}} -$$

$$- \underbrace{\partial^\mu j_\mu}_{\substack{=0 \text{ current} \\ \text{conservation}}} \Lambda = j_\mu A^\mu$$

$\Rightarrow j_\mu A^\mu$ is gauge-invariant for conserved current j_μ.

$\Rightarrow$ need to find a gauge in which to look for solution: possible gauges are

(i) Lorentz gauge $\partial_\mu A^\mu = 0$

(ii) Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$

Work in covariant gauge $\partial_\mu A^\mu = 0 \Rightarrow$ Maxwell equations become $\square A^\mu = j^\mu$, $\Rightarrow$ put $j^\mu = 0 \Rightarrow$ get $\boxed{\square A^\mu = 0}$

Write the solution as:

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k} \sum_{\lambda=\pm, L} \Big[\hat{a}_{\vec{k},\lambda}\, \varepsilon_\mu^{(\lambda)}(k)\, e^{-ik\cdot x} + $$
$$+ \hat{a}^{+}_{\vec{k},\lambda}\, \varepsilon_\mu^{(\lambda)*}\, e^{ik\cdot x} \Big]$$

gauge condition $\partial_\mu A^\mu = 0 \Rightarrow k_\mu \varepsilon^{(\lambda)\mu}(k) = 0$

$\Rightarrow$ if $k^\mu = (\underset{t}{k}, \underset{x}{0}, \underset{y}{0}, \underset{z}{k}) \Rightarrow$

$\varepsilon_\mu^{\pm}(k) = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$, $\varepsilon_\mu^{L} = (1, 0, 0, 1)$ work.

Quantize by requiring $\left[\hat{a}_{\vec{k},\lambda}, \hat{a}^{+}_{\vec{k}',\lambda'}\right] = -g_{\lambda\lambda'}\, \delta(\vec{k}-\vec{k}')\cdot (2\pi)^3\, 2\varepsilon_k$

note that

$\varepsilon^{(\lambda)}\cdot\varepsilon^{*(\lambda')} = g^{\lambda\lambda'}$

for $\lambda = \pm$.

$\varepsilon^L\cdot\varepsilon^{*L} = 0 \Rightarrow$ zero probability.

$\varepsilon_k = |\vec{k}|$

as photons have zero mass

The Hamiltonian: $$\boxed{H = \int \frac{d^3k}{(2\pi)^3\, 2\mathcal{E}_k}\, \mathcal{E}_k \sum_{\lambda=\pm} a^+_{\vec{k},\lambda}\, a_{\vec{k},\lambda}}$$

$\rightarrow$ no longitudinal photon contribution to Hamiltonian and, therefore, time-evolution.

$\Rightarrow$ proper way is to require that all physical states have $\partial_\mu A^{\mu\,(+)} |\psi\rangle = 0$ ← positive frequencies while quantizing by adding a term like $\lambda(\partial_\mu A^\mu)^2$ to the Hamiltonian. (a constraint)

Time evolution: $\boxed{-i\frac{\partial}{\partial t} A_\mu = [H, A_\mu]}$ as usual.

Quark Model and Group Theory

Quarks.

Many meson & baryon resonances were discovered in the 1960's. People wanted to organize the data: they started noticing some pattern.

Take p & n (known from 1932): both are nucleons with spin ½. They have almost identical masses: $M_p \simeq 938\,\text{MeV}$, $M_n = 940\,\text{MeV}$. Proton has charge $+e$, neutron has charge 0.