Last time: we talked about gauge fields A^μ:

the lagrangian is $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu$

equations of motion: $\partial_\nu F^{\mu\nu} = -j^\mu$

there are Maxwell equations

Lagrangian is invariant under gauge transform:

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \Lambda$$

$\Rightarrow$ pick a gauge: we chose Lorenz gauge $\partial_\mu A^\mu = 0$

$\Rightarrow$ Maxwell eqn's become $\boxed{\Box A^\mu = 0}$ in vacuum. (no sources)

The solution is
$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \sum_{\lambda=\pm,L} \left[\hat{a}_{\vec{k},\lambda} \, \varepsilon_\mu^{(\lambda)}(k) \, e^{-ik\cdot x} + \hat{a}^\dagger_{\vec{k},\lambda} \, \varepsilon_\mu^{(\lambda)*} e^{ik\cdot x} \right]$$

with $k\cdot\varepsilon^{(\lambda)} = 0$; $\varepsilon_K = |\vec{k}|$.

~ Quantize by promoting a's into operators.

$$H = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_K} \varepsilon_K \sum_{\lambda=\pm} \hat{a}^\dagger_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda}$$

~ only transverse polarizations are physical

Time evolution: $i \dfrac{\partial A_\mu}{\partial t} = [A_\mu, H]$

The Hamiltonian: $$H = \int \frac{d^3k}{(2\pi)^3\, 2\mathcal{E}_k}\, \mathcal{E}_k \sum_{\lambda=\pm} \hat{a}^+_{\vec{k},\lambda}\, \hat{a}_{\vec{k},\lambda}$$

$\Rightarrow$ no longitudinal photon contribution to Hamiltonian and, therefore, time-evolution.

$\Rightarrow$ proper way is to require that all physical states have $\partial_\mu A^{\mu\,(+)} |\psi\rangle = 0$ ← positive frequencies while quantizing by adding a term like $\lambda(\partial_\mu A^\mu)^2$ to the Hamiltonian. (a constraint)

Time evolution: $$-i \frac{\partial}{\partial t} A_\mu = [H, A_\mu]$$ as usual.

Quark Model and Group Theory

Quarks.

Many meson & baryon resonances were discovered in the 1960's. People wanted to organize the data: they started noticing some pattern.

Take p & n (known from 1932): both are nucleons with spin $1/2$. They have almost identical masses: $M_p \simeq 938$ MeV, $M_n = 940$ MeV. Proton has charge $+e$, neutron has charge 0.

Other similar groups of particles were found:

pions are π^+, π^-, π^0 ~ spin-0 mesons.

$m_{\pi^\pm} = 140\ MeV$, $m_{\pi^0} = 136\ MeV$

rho-mesons: ρ^+, ρ^0, ρ^- ~ spin-1 mesons, $m_\rho = 770\,MeV$

Δ-baryons: $\Delta^{++}, \Delta^+, \Delta^0, \Delta^-$ ~ spin-$3/2$ baryons

$m_\Delta \simeq 1232\,MeV$

~ to organize these introduce a new quantum number called isospin. Works like the angular momentum operator in the "isospin space". If $\vec{I}$ is the isospin operator =>

=> can construct $\vec{I}^2$ & I_z:

$\vec{I}^2$ has eigenvalues $I(I+1)$, $I = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

I_z — " — $-I, -I+1, \ldots, +I$ =>

=> multiplicity $= 2I+1$.

=> can classify hadrons as states $|I, I_3\rangle$:

$$\begin{array}{ll} \text{proton: } & p = |\overset{I}{\tfrac{1}{2}}, \overset{I_3}{+\tfrac{1}{2}}\rangle \\ \text{neutron: } & n = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \end{array} \Rightarrow N = \begin{pmatrix} p \\ n \end{pmatrix} \text{ iso-doublet}$$

(neglect mass difference) $2I+1 = 2 \Rightarrow I = 1/2$.

Pions: $2I+1=3 \Rightarrow I=1 \Rightarrow$

$$\left.\begin{aligned} \pi^+ &= |\overset{I}{1}, \overset{I_3}{1}\rangle \\ \pi^0 &= |1, 0\rangle \\ \pi^- &= |1, -1\rangle \end{aligned}\right\} \text{ iso-triplet!}$$

ρ-mesons:

$$\left.\begin{aligned} \rho^+ &= |1, 1\rangle \\ \rho^0 &= |1, 0\rangle \\ \rho^- &= |1, -1\rangle \end{aligned}\right\} \begin{aligned} &\text{iso-triplet!} \\ &\text{(now spin-1)} \end{aligned}$$

Δ-baryon: $2I+1=4 \Rightarrow I=3/2 \Rightarrow$

$$\left.\begin{aligned} \Delta^{++} &= |\overset{I}{\tfrac{3}{2}}, \overset{I_3}{\tfrac{3}{2}}\rangle \\ \Delta^+ &= |\tfrac{3}{2}, \tfrac{1}{2}\rangle \\ \Delta^0 &= |\tfrac{3}{2}, -\tfrac{1}{2}\rangle \\ \Delta^- &= |\tfrac{3}{2}, -\tfrac{3}{2}\rangle \end{aligned}\right\} \text{ 4-multiplet}$$

$\Rightarrow$ different <u>representations</u> of <u>group</u> SU(2) isospin.

(we'll discuss groups shortly)

Example:

Consider 2 processes: (i) $p + p \to d + \pi^+$
(ii) $p + n \to d + \pi^0$

(i) LHS: $|\tfrac{1}{2}, \tfrac{1}{2}\rangle |\tfrac{1}{2}, \tfrac{1}{2}\rangle = |1, 1\rangle$ ← total isospin state

RHS: deuteron has isospin $= 0 \Rightarrow \pi^+ = |1,1\rangle$

$\Rightarrow$ amplitude $\propto \langle 1,1 | 1,1\rangle \sim 1$.

(ii) LHS: $|\tfrac{1}{2}, \tfrac{1}{2}\rangle |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,0\rangle\right)$

as
$$\left.\begin{aligned} |1,0\rangle &= \frac{1}{\sqrt{2}}\left(|p\rangle|n\rangle + |n\rangle|p\rangle\right) \\ |0,0\rangle &= \frac{1}{\sqrt{2}}\left(|p\rangle|n\rangle - |n\rangle|p\rangle\right) \end{aligned}\right\} \Rightarrow |p\rangle|n\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,0\rangle\right)$$

RHS: $\pi^0 = |1,0\rangle \Rightarrow$

amplitude $\propto \frac{1}{\sqrt{2}}\left(\langle 1,0| + \langle 0,0|\right)|1,0\rangle \sim \frac{1}{\sqrt{2}}$

$$\Rightarrow \quad \frac{\sigma_{pp\to d\pi^+}}{\sigma_{pn\to d\pi^0}} \propto \frac{|M_{pp\to d\pi^+}|^2}{|M_{pn\to d\pi^0}|^2} = \frac{1}{1/2} = 2.$$

~ agrees with experiment!

~ all given by Clebsch-Gordan coefficients.

Note that for pions & ρ's: $Q = I_3$

where Q is the electric charge. (in units of $|e|$)

For Δ's: $Q = I_3 + \frac{1}{2}$, same for protons!

Def. Baryon number ~ a quantum number counting the # of baryons: $B = +1$ for baryons, $B = -1$ for anti-baryons.

$$\Rightarrow \quad Q = I_3 + \frac{B}{2}$$

However, there are also kaons:

(can see by looking at decay)

K^+, K^0 form an isospin-doublet $\Rightarrow I = 1/2 \Rightarrow$

$\Rightarrow K^+ = |\frac{1}{2}, \frac{1}{2}\rangle$, $K^0 = |\frac{1}{2}, -\frac{1}{2}\rangle$ $\Rightarrow$ $B = 0$ (mesons, spin-0)

$\Rightarrow$ would expect $Q = I_3$, but really only

$Q = I_3 + \frac{1}{2}$ works.

$K^-, \bar{K}^0$ are also spin-0 mesons, also form a doublet

$\bar{K}^0 = |\frac{1}{2}, \frac{1}{2}\rangle$, $K^- = |\frac{1}{2}, -\frac{1}{2}\rangle$ $\Rightarrow$ $Q = I_3 - \frac{1}{2}$ now...

Def. A new quantum number of strangeness:

for mesons $Q = I_3 + \frac{S}{2}$ $\Rightarrow$ K^+, K^0 have $S = +1$

$\bar{K}^0, K^-$ have $S = -1$.

Elementary particles can be characterized by their quantum number using J^{PC} notation.

J ~ the spin of a particle

P ~ <u>parity</u>: $\mathbb{P}: \vec{x} \to -\vec{x}$

$P|\psi\rangle = K|\psi\rangle \Rightarrow P^2|\psi\rangle = |\psi\rangle = K^2|\psi\rangle \Rightarrow K = \pm 1$

all particles have definite parity $= \pm 1$.

<u>mesons</u>:

pions: $P = -1$

ρ's: $P = -1$

<u>baryons</u>: > proton $P = +1$

C ~ <u>charge conjugation</u>: C transforms particles into anti-particles (particles with the same mass and opposite quantum numbers, e.g. holes in Dirac sea)

$C|\psi\rangle = \eta|\bar{\psi}\rangle \Rightarrow C^2|\psi\rangle = \eta^2|\psi\rangle = |\psi\rangle$

$\Rightarrow \eta^2 = 1 \Rightarrow \eta = \pm 1$.

$C|\gamma\rangle = -|\gamma\rangle$ $\quad |\gamma\rangle$ is a photon (change charge from $+ \to -$ $\Rightarrow$ get a $-$ sign)

$\Rightarrow$ as $\pi^0 \to \gamma\gamma \Rightarrow C|\pi^0\rangle = |\pi^0\rangle \Rightarrow C = +1$ for π^0.

~ not all particles are eigenstates of C. (e.g. $\pi^\pm$ are <u>not</u>!)

$\Rightarrow$ proton: $J = \frac{1}{2}$, $P = +1$ $\Rightarrow$ $\left(\frac{1}{2}\right)^+$

Δ : $J = \frac{3}{2}$, $P = +1$ $\Rightarrow$ $\frac{3}{2}^+$

pion (π^0): $J = 0$, $P = -1$, $C = +1$ $\Rightarrow$ 0^{-+}

$\pi^\pm$: $J = 0$, $P = -1$ $\Rightarrow$ 0^-

ρ : $J = 1$, $P = -1$ $\Rightarrow$ 1^{--}
$C = -1$

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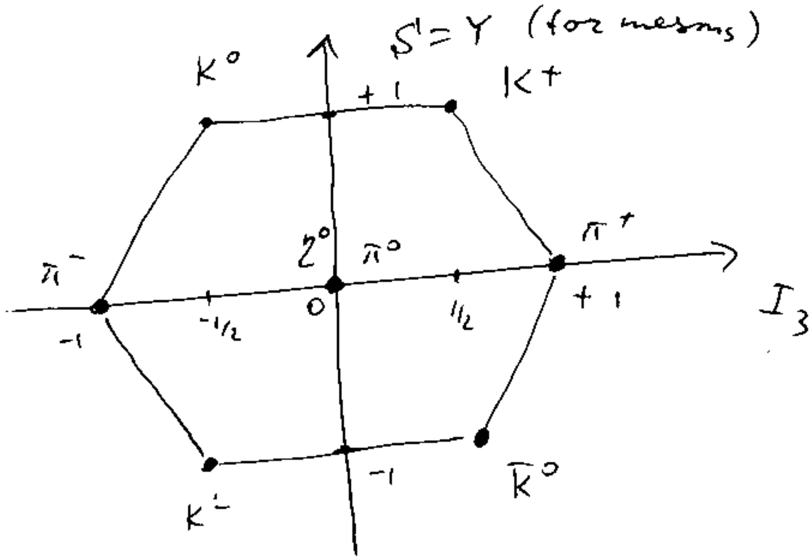
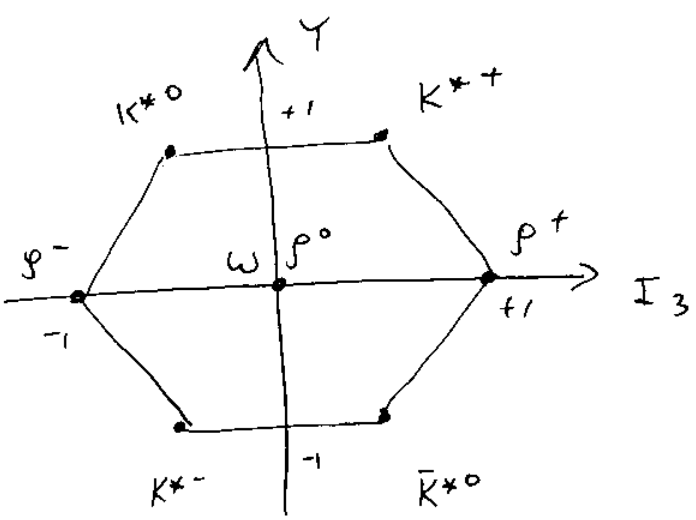
general formula: $Q = I_3 + \frac{B}{2} + \frac{S}{2}$. Gell-Mann Nishijima ('53, '55)

$Y = B + S$ $\Rightarrow$ $\boxed{Q = I_3 + \frac{Y}{2}}$, Y ~ hypercharge

Gell-Mann & Ne'eman (1961): take, say, all 0^- mesons: π's, K's, η :

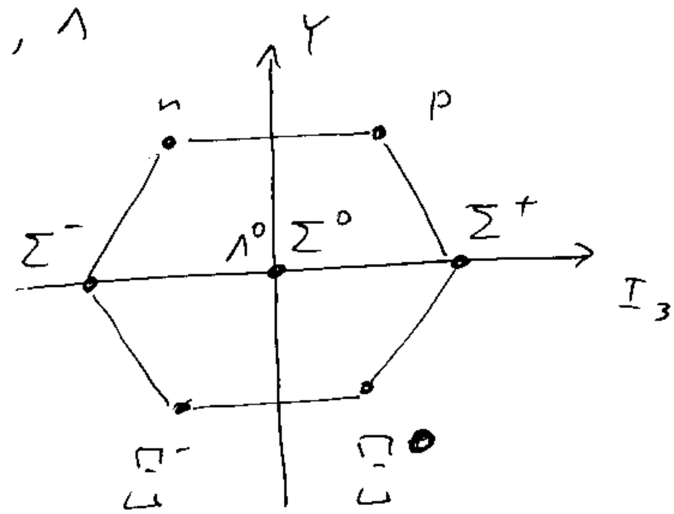


spin-1 mesons (1^-):



The "Eightfold Way".

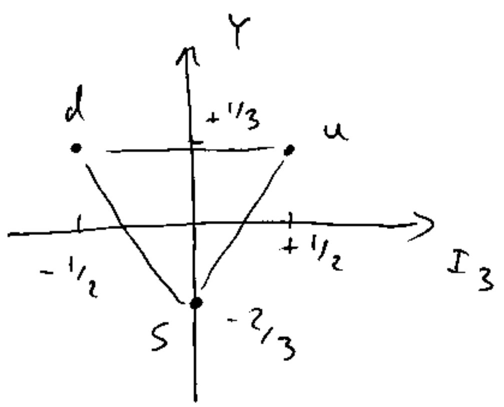
Baryons: $\frac{1}{2}^+$: $p, n, \Sigma, \Xi, \Lambda$



$\Rightarrow$ there must be some sub-structure for all these symmetries.

$\Rightarrow$ Gell-Mann & Ne'eman suggested that there exist

3 quarks: u, d, s (up, down, strange)
spin-$1/2$ particles.



$B = +\frac{1}{3}$ for each quark

$I_3 = +\frac{1}{2}$ for u, $-\frac{1}{2}$ for d, 0 for s

$S = 0$ for u, d, $S = -1$ for strange

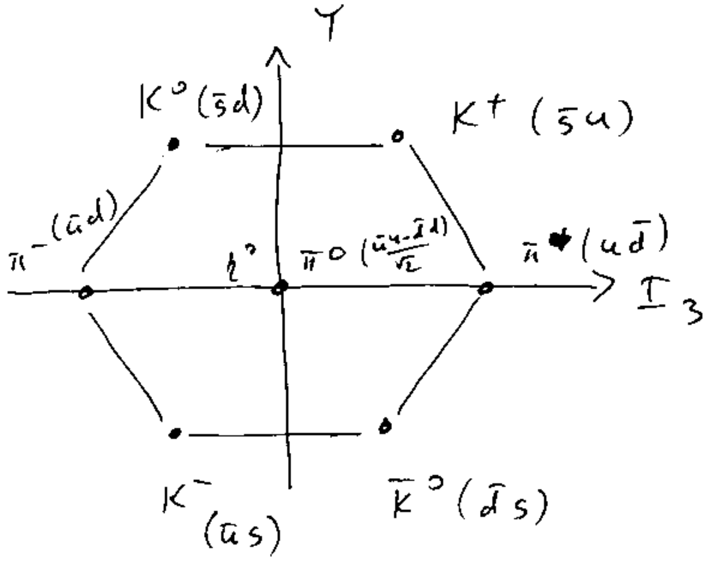
$Y = B + S \Rightarrow Y = +\frac{1}{3}$, u, d, $-\frac{2}{3}$, for s

$\Rightarrow$ if $\pi^+ = u\bar{d}$, $\pi^- = \bar{u}d$, $\pi^0 = \frac{1}{\sqrt{2}}(\bar{u}u - \bar{d}d)$

$K^+ = \bar{s}u$, $K^0 = \bar{s}d$, $\bar{K}^0 = \bar{d}s$, $K^- = \bar{u}s$

$\eta^0 = \frac{1}{\sqrt{6}}(\bar{u}u + \bar{d}d - 2\bar{s}s)$

$\pi^+ = u\bar{d} \Rightarrow Y = B + S = \frac{1}{3} - \frac{1}{3} = 0$; $I_3 = +\frac{1}{2} - (-\frac{1}{2}) = 1$.



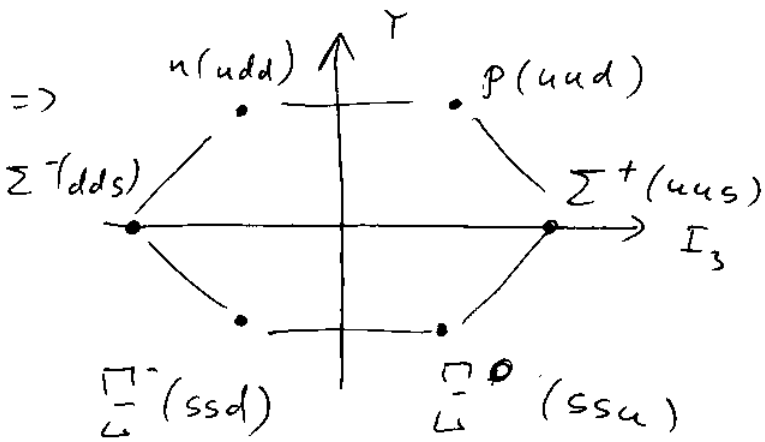
$K^0 = \bar{s}d \Rightarrow Y = 1$,

$I_3 = \frac{1}{2}$, etc.

baryons: $p \sim uud$, $n \sim udd$, $\Sigma^+ = uus$, $\Sigma^- = dds$,

$\Sigma^0 = s\,\frac{ud+du}{\sqrt{2}}$, $\Xi^0 \sim ssu$, $\Xi^- = ssd$, $\Lambda^0 \sim s\,\frac{ud-du}{\sqrt{2}}$

$\Rightarrow$



e.g. $p(uud) \Rightarrow$

$Y = B + S = +\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = +1$

$I_3 = +\frac{1}{2} + \frac{1}{2} - \frac{1}{2} = \frac{1}{2}$.

etc.

$\Rightarrow$ in reality there are 6 quark flavors:

u, d, s, c, b, t

c: charm, b: bottom, t: top

$c, b, t \sim$ heavier, do not contribute to the lightest hadrons considered so far.