Last time: talked about hadrons & defined:

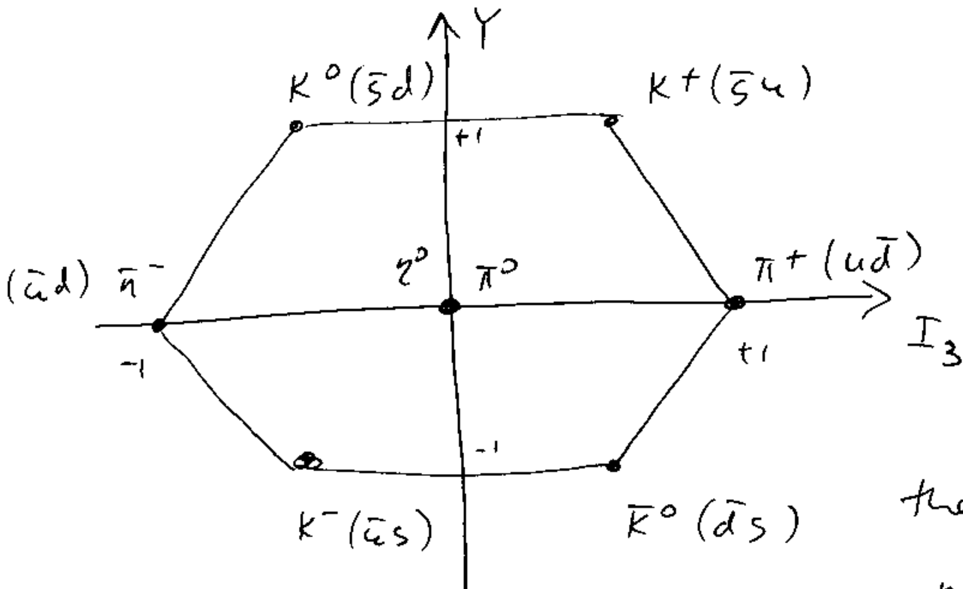
- Isospin $\vec{I}$ with I_3 the isospin projection
- Baryon number B ~ # of baryons − # of anti-baryons
- Strangeness: K^+, K^0 have $S = +1$, $\bar{K}^0, K^-$ have $S = -1$.
- Hypercharge $Y = B + S$

We observed that $Q = I_3 + \frac{Y}{2}$ for all known hadrons.

(Gell-Mann, Nishijima)

- Introduced J^{PC} classification.

- Gell-Mann & Ne'eman ('61) found the "Eightfold Way":



0^- mesons

(pseudoscalar mesons)

$$\pi^0 = \frac{1}{\sqrt{2}}(\bar{u}u - \bar{d}d)$$

$$\eta^0 = \frac{1}{\sqrt{6}}(\bar{u}u + \bar{d}d - 2\bar{s}s)$$

there is also

$$\eta' = \frac{1}{\sqrt{3}}(\bar{u}u + \bar{d}d + \bar{s}s)$$

~ but η' is a flavor-singlet particle ⇒

⇒ different representation of $SU(3)$ flavor ⇒

⇒ not shown here (also has a mass of 958 MeV ~

~ much heavier)

1^* (vector meson) ~ same, $\rho^0 = \frac{1}{\sqrt{2}}(\bar{u}u - \bar{d}d)$, $\omega^0 = \frac{1}{\sqrt{6}}(\bar{u}u + \bar{d}d - 2\bar{s}s)$

- there is also a singlet $\phi^0 = \frac{1}{\sqrt{3}}(\bar{u}u + \bar{d}d + \bar{s}s)$

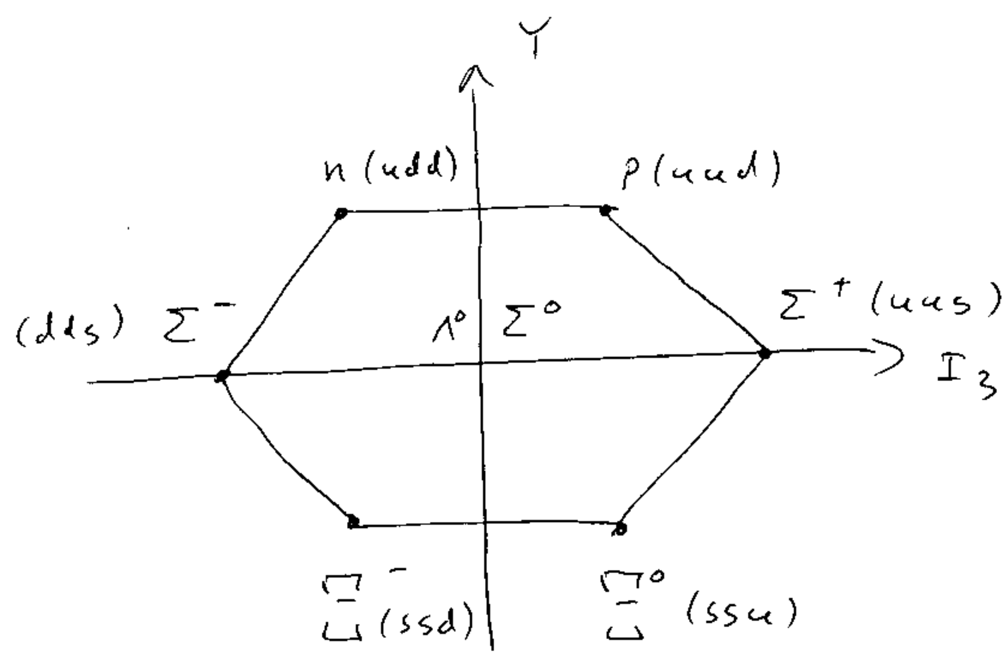
- but ϕ^0 (flavor singlet) & ω^0 (flavor octet) mix strongly, such that the physical states are

$$\phi = \frac{1}{\sqrt{3}}(\phi^0 - \sqrt{2}\,\omega^0) = s\bar{s}$$

$$\omega = \frac{1}{\sqrt{3}}(\omega^0 + \sqrt{2}\,\phi^0) = \frac{u\bar{u} + d\bar{d}}{\sqrt{2}}$$

(this is not essential, just FYI)

- baryons work too: $\frac{1}{2}^+$:



$$\Sigma^0 = s\,\frac{ud + du}{\sqrt{2}}$$

$$\Lambda^0 = s\,\frac{ud - du}{\sqrt{2}}$$

What about spin-$\frac{3}{2}$ baryons?

$\frac{3}{2}^+$ baryons form a decuplet:

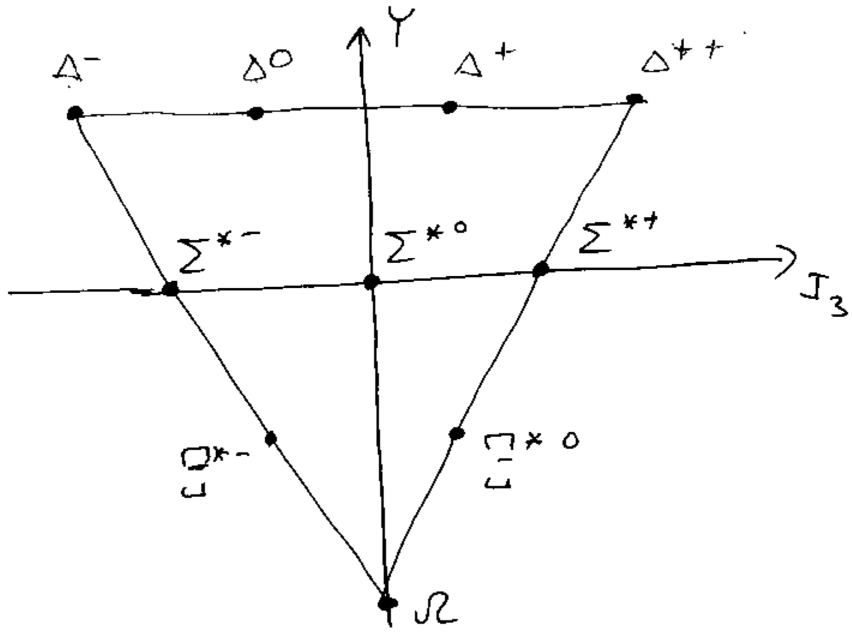
$\Delta^{++} = uuu$, $\Delta^+ \sim uud$,

$\Delta^0 \sim udd$, $\Delta^- \sim ddd$

$\Sigma^{*+} \sim suu$, $\Sigma^{*0} \sim sud$,

$\Sigma^{*-} \sim sdd$, $\Xi^{*0} \sim ssu$,

$\Xi^{*-} \sim ssd$, $\Omega^- \sim sss$.



$\Rightarrow$ seems OK? but let's look at Δ^{++} for instance: it has spin-$3/2$ $\Rightarrow$ the spin state is

$|\underset{u\,u\,u}{\uparrow\uparrow\uparrow}\rangle = |\uparrow\rangle \otimes |\uparrow\rangle \otimes |\uparrow\rangle \Rightarrow$ symmetric

the isospin state is $|\uparrow\uparrow\uparrow\rangle$ too ~ also symmetric!

What happens to Pauli principle ~ the full wave function $\Psi = \Psi_{spin} \otimes \Psi_{isospin}$ has to be anti-symmetric! (Fermi-Dirac statistic)

$\Psi_{spatial}$ is symmetric too (ground state for uuu)

$\Rightarrow$ the way out is to postulate a new quantum number called color (Greenberg, Han, Nambu '64-'66)

there are 3 colors: $i = 1, 2, 3 \Rightarrow u_i(x)$ ~ up quark w.f.

$\Rightarrow \Delta^{++} \propto \varepsilon^{ijk}\, u_i(x_1)\, u_j(x_2)\, u_k(x_3)$

anti-symmetric.

$\Rightarrow$ let us summarize our knowledge about quarks in a table:

	Q	B	I	I_3	S	mass (bare)
u (up)	$+2/3$	$1/3$	$\frac{1}{2}$	$\frac{1}{2}$	0	1,5 – 3,3 MeV
d (down)	$-1/3$	$1/3$	$\frac{1}{2}$	$-\frac{1}{2}$	0	3,5 – 6,0 MeV
s (strange)	$-1/3$	$1/3$	x	x	-1	104^{+26}_{-34} MeV
c (charm)	$+2/3$	$1/3$	x	x	0	1,27 GeV
b (bottom)	$-1/3$	$1/3$	x	x	0	4,2 GeV
t (top)	$+2/3$	$1/3$	x	x	0	171 GeV

All quantum numbers flip signs for anti-quarks: $\bar{u}, \bar{d}, \bar{s}, \bar{c}, \bar{b}, \bar{t}$. (Baryon # of $\bar{u}$ is $-1/3$, e.g.)

All quarks are fermions $\Rightarrow$ should be described by Dirac spinors $q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$

$\Rightarrow q_\alpha,\ \alpha = 1,2,3,4$ would be correct.

But: what about different colors & flavors?

$\Rightarrow$ write q_α^{af} — color index, $a = 1, 2, 3$; flavor index $f = u, d, s, c, b, t$; spinor index

$\Rightarrow$ quark Lagrangian is

$$\boxed{\mathcal{L}_{quarks} = \bar{q}^{af} (i\gamma\cdot\partial - m_f) q^{af}}$$

~ sum over a, f implied, m_f ~ bare quark masses

$\Rightarrow$ is that it for strong interactions?

No, quarks should be able to interact with each other!

$\Rightarrow$ one needs <u>gluons</u>: $A_\mu^i(x)$, $i = 1, \ldots, 8$ ~ 8 different gauge fields

$$\boxed{\mathcal{L}_{gluons} = -\frac{1}{4} F_{\mu\nu}^i F^{i\mu\nu}}$$

with $F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g f_{ijk} A_\mu^j A_\nu^k$

g ~ gluon self-coupling constant, f_{ijk} ~ structure constants of $SU(3)$

$\Rightarrow$ What about quark-gluon interactions?

$$\boxed{\mathcal{L}_{int} = g\, \bar{q}^{a'f} \gamma^\mu A_\mu^i (T^i)_{a'a} q^{af}}$$

, where $(T^i)_{ab}$ are 3×3 matrices (generators of $SU(3)$), $i = 1, \ldots, 8$, $a, b = 1, 2, 3$

⇒ putting all this together write the lagrangian for Quantum Chromodynamics (QCD) – the theory of strong interactions:

$$\boxed{\begin{aligned}\mathcal{L}_{QCD} &= \bar{q}^{af}(i\gamma\cdot\partial - m_f)q^{af} - \frac{1}{4}F^i_{\mu\nu}F^{\mu\nu i} \\ &\quad + g\,\bar{q}^{bf}\gamma^\mu A^i_\mu (T^i)_{ba}\, q^{af}\end{aligned}}$$

<u>Elements of Group Theory</u>

Def. A <u>Group</u> G is a set of elements with a multiplication law having the following properties:

(i) Closure: if $f, g \in G \Rightarrow h = f\cdot g \in G$

(ii) Associativity: $f, g, h \in G \Rightarrow f\cdot(g\cdot h) = (f\cdot g)\cdot h$

(iii) Identity: $\exists\, e \in G \;\; \forall f \in G: ef = fe = f$

(iv) Inverse element: $\forall f \in G\; \exists f^{-1} \in G: ff^{-1} = f^{-1}f = e$.

<u>Example</u>: $\{1, e^{i\frac{\pi}{2}}, e^{i\pi}, e^{i\frac{3}{2}\pi}\}$ form a group (why?). $\mathbb{Z}_4$ " $\{1, i, -1, -i\}$.

Integers: $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ form a group. What is e there? Def. $H \subset G \Rightarrow H$ is a <u>subgroup</u>.

Def. A group is called Abelian if for any

$f, g \in G$: $f \cdot g = g \cdot f$

otherwise it is called non-Abelian ($f \cdot g \neq g \cdot f$)

Example (important!) $n \times n$ unitary matrices form a group: $U U^+ = U^+ U = \mathbb{1}$ (unitary matrices).

Def. Such group is denoted $U(n)$, ($e = \mathbb{1} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$)
and is called the unitary group.

Sub-example) $U(1)$: 1×1 matrices $\Rightarrow e^{i\varphi}$, φ ~ #

$\varphi \in \mathbb{R}$ ~ form a group, $e = 1$.

Def. $n \times n$ unitary matrices with unit determinant

($U U^+ = U^+ U = \mathbb{1}$, $\det U = +1$) form a group too!

It is called special unitary group and is denoted

$SU(n)$.

Def. A representation of group G is a mapping D of group elements: $f \in G$: $f \to D(f)$, where $D(f)$ is a space of linear operators (e.g. matrices) such that:

(i) $D(e) = \mathbb{1}$

(ii) $D(g_1) D(g_2) = D(g_1 g_2)$ for $g_1, g_2 \in G$.

Take a group $\mathbb{Z}_4$: it has $\{e, g_1, g_2, g_3\}$

Our example $\{1, e^{i\frac{\pi}{2}}, e^{i\pi}, e^{i\frac{3\pi}{2}}\} = \{D(e), D(g_1), D(g_2), D(g_3)\}$

is one of the many possible representations of $\mathbb{Z}_4$.

Def. Dimension of representation is the dimension of the space of D-matrices.

Def. Representation is called reducible if $\exists M$ (matrix) such that

$$M\, D(g)\, M^{-1} = \begin{bmatrix} D_1(g) & 0 & \\ 0 & D_2(g) & \\ & & \ddots \end{bmatrix} \quad \text{for } \forall g \in G.$$

$$\Rightarrow D = D_1 \oplus D_2 \oplus \ldots$$

a representation is called irreducible if no such matrix M exists.

Def. For two groups $G = \{g_1, g_2, \ldots\}$, $H = \{h_1, h_2, \ldots\}$ define direct-product group $G \otimes H = \{g_i h_j\}$

such that $g_k h_e \cdot g_m h_n = g_k g_m \cdot h_e h_n$.

Lie Groups

Imagine a group G with elements smoothly dependent on a continuous set of parameters α_i, $i = 1, \ldots, N$: $g(\alpha_i) \in G$.