<u>Last time:</u> ~ defined <u>color</u> of quarks

Wrote out the QCD (Quantum Chromodynamics) lagrangian:

$$\mathcal{L}_{QCD} = \bar{q}^{af}(i\gamma\cdot\partial - m_f)\, q^{af} - \frac{1}{4} F^{i}_{\mu\nu} F^{i\mu\nu} + g\,\bar{q}^{bf}\gamma^{\mu} A^{i}_{\mu} (T^{i})_{ba}\, q^{af}$$

with T^i the generators of $SU(3)$, A^i_μ ~ gluon fields $i=1,\dots,8$

<u>Elements of Group Theory</u> (cont'd)

~ We defined a group G: (i) $f\cdot g = h \in G$, (ii) $f(gh) = (fg)h$
(iii) $\exists e$: $f.e = e.f = f$ (i) f^{-1}.

~ Abelian: $f\cdot g = g\cdot f$, non-Abelian $[f, g] \neq 0$.

$U(N)$: $N\times N$ unitary matrices $\Rightarrow$ unitary group

$SU(N)$: — " — $\oplus$ $\det U = +1$. $\Rightarrow$ special — " —

~ Representation: $f \to D(f)$: (i) $D(e) = \mathbb{1}$
(ii) $D(g_1 g_2) = D(g_1) D(g_2)$

D ~ matrices $\Rightarrow$ dim. of representation is their size.

Take a group $\mathbb{Z}_4$: it has $\{e, g_1, g_2, g_3\}$

Our example $\{1, e^{i\frac{\pi}{2}}, e^{i\pi}, e^{i\frac{3\pi}{2}}\} = \{D(e), D(g_1), D(g_2), D(g_3)\}$

is one of the many possible representations of $\mathbb{Z}_4$.

Def. Dimension of representation is the dimension of the space of D-matrices.

Def. Representation is called reducible if $\exists M$ (matrix) such that

$$M\, D(g)\, M^{-1} = \begin{bmatrix} D_1(g) & 0 & \\ 0 & D_2(g) & \\ & & \ddots \end{bmatrix} \quad \text{for } \forall g \in G.$$

$\Rightarrow D = D_1 \oplus D_2 \oplus \ldots$

a representation is called irreducible if no such matrix M exists.

Def. For two groups $G = \{g_1, g_2, \ldots\}$, $H = \{h_1, h_2, \ldots\}$ define direct-product group $G \times H = \{g_i h_j\}$

such that $g_k h_l \cdot g_m h_n = g_k g_m \cdot h_l h_n$.

Lie Groups

Imagine a group G with elements smoothly dependent on a continuous set of parameters α_i, $i = 1, \ldots, N$: $g(\alpha_i) \in G$.

$\Rightarrow$ assume that $g(\alpha_i = 0) = e$ (the identity element)

$\Rightarrow$ for a representation of the group:

$$D(\alpha_i = 0) = \mathbb{1}.$$

Taylor expand $D(\alpha_i)$ near 0:

$$D(\delta\alpha_i) = \mathbb{1} + i\,\delta\alpha_i X_i + \ldots = \mathbb{1} + i\,\delta\vec{\alpha}\cdot\vec{X}$$

(summation over repeating indeces assumed)

Def. X_i are called <u>generators</u> of the group.

$\delta\vec{\alpha} = \frac{\vec{\alpha}}{k}$, k - integer

$$D(\alpha_i) = D(\delta\alpha_i)\,D(\delta\alpha_i)\ldots D(\delta\alpha_i) = \lim_{k\to\infty}\left(\mathbb{1} + i\,\delta\vec{\alpha}\cdot\vec{X}\right)^k$$

$$= \lim_{k\to\infty}\left(\mathbb{1} + i\,\frac{\vec{\alpha}}{k}\cdot\vec{X}\right)^k = e^{i\vec{\alpha}\cdot\vec{X}}.$$

Def. A group with elements depending smoothly on continuous set of parameters α_i, $i = 1, \ldots, N$, with generators X_i is called a <u>Lie group</u>.

$D(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{X}}$ $\Rightarrow$ as D can be a matrix

$\Rightarrow$ $\vec{X}$ can be a matrix; therefore in general $[X_i, X_j]$ does not have to be 0.

$\Rightarrow$ however $D(\vec{\alpha})\,D(\vec{\beta}) = e^{i\vec{\alpha}\cdot\vec{X}}\, e^{i\vec{\beta}\cdot\vec{X}}$ is

also a group element $\Rightarrow e^{i\vec{\alpha}\cdot\vec{X}}\, e^{i\vec{\beta}\cdot\vec{X}} = e^{i\vec{\delta}\cdot\vec{X}}$

$\Rightarrow$ can show that for this to work we need

$$[X_a, X_b] = i f_{abc} X_c$$

Lie algebra

f_{abc} ~ structure constants of the group

$$f_{abc} = - f_{bac}.$$

f_{abc} are <u>real</u> for unitary representations (for hermitean X_a).

<u>Example</u> take the group $SU(2)$: unitary 2×2 matrices with $\det = +1$ $(UU^+ = U^+U = 1, \det U = +1)$.
(defining representation)
Using Pauli matrices we can define a representation of $SU(2)$:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\Rightarrow D(\vec{\alpha}) = e^{i\frac{\vec{\alpha}\cdot\vec{\sigma}}{2}}$, $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ a 3-vector.

rotations around $\frac{\vec{\alpha}}{|\vec{\alpha}|}$ axis by angle $|\vec{\alpha}|$.

as $\sigma_i^+ = \sigma_i$ (hermitean) => any 2×2 unitary matrix with det = +1 can be represented as $e^{i \frac{\vec{\alpha} \cdot \vec{\sigma}}{2}} = U$.

Check: $U U^+ = e^{i \frac{\vec{\alpha} \cdot \vec{\sigma}}{2}} e^{-i \frac{\vec{\alpha} \cdot \vec{\sigma}}{2}} = \mathbb{1}$.

$\det U = \det e^{i \frac{\vec{\alpha} \cdot \vec{\sigma}}{2}} = \Big|$ as $\det e^A = e^{\operatorname{tr} A}$ $= 1$

as $\operatorname{tr} \sigma_i = 0$.

=> there are $2^2 - 1 = 3$ different (linearly independent) $n \times n$ traceless hermitean matrices => $\{\sigma_i\}$ use up all possibilities.

Generators: $J_i = \frac{\sigma_i}{2}$ => $D(\vec{\alpha}) = e^{i \vec{\alpha} \cdot \vec{J}}$.

=> SU(2) is a Lie group

We know that $[\sigma_i, \sigma_j] = 2i \, \varepsilon_{ijk} \sigma_k$ =>

=> $[J_i, J_j] = i \, \varepsilon_{ijk} J_k$

=> generators of SU(2) form a Lie algebra with structure constants ε_{ijk}

ε_{ijk} : totally anti-symmetric Levi-Civita symbol, $\varepsilon_{123} = 1$, $\varepsilon_{ijk} = -\varepsilon_{jik} = \varepsilon_{jki}$ ...

$\varepsilon_{112} = 0$ ....

<u>Another example:</u> $SU(3)$: 3×3 unitary matrices with $\det = +1$

Define Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{(cf. Pauli matrices)}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Normalization convention $\mathrm{tr}[\lambda_i \lambda_j] = 2\,\delta_{ij}$.

There are $3^2 - 1 = 8$ traceless hermitean 3×3 matrices

$\Rightarrow$ these should work.

Generators of $SU(3)$: $T^a = \dfrac{\lambda^a}{2}$ $\Rightarrow$

$\Rightarrow [T^a, T^b] = i f^{abc}\, T^c$, with structure constants f^{abc}, which are anti-symmetric under the interchange of any two indeces.

$\Rightarrow SU(3)$ is a Lie group with the generator algebra given above.

a	b	c	f^{abc}
1	2	3	1
1	4	7	1/2
1	5	6	−1/2
2	4	6	1/2
2	5	7	1/2
3	4	5	1/2
3	6	7	−1/2
4	5	8	$\sqrt{3}/2$
6	7	8	$\sqrt{3}/2$

$f_{112} = 0$ ...

all other f^{abc}'s can be obtained from this table.

Casimir operator: commutes with all generators:

$\vec{T}^2 = T_1^2 + T_2^2 + \ldots + T_n^2 = \frac{N^2-1}{2N}$

$\Rightarrow$ for $SU(2)$ it is $3/4$

for $SU(3)$ it is $4/3$.

$$D(\vec{A}) = e^{i\vec{A}\cdot\vec{T}}, \text{ with } \vec{A} = (A_1, A_2, \ldots, A_8)$$

~ an 8-component vector.

Jacobi Identity and the Adjoint Representation

~ go back to some general Lie group with the generators X_a obeying some Lie algebra

$$[X_a, X_b] = i f_{abc} X_c.$$

One can then easily prove Jacobi identity:

$$[X_a,[X_b,X_c]]+[X_b,[X_c,X_a]]+[X_c,[X_a,X_b]]=0.$$

(prove this by using definitions of commutators)

$\Rightarrow$ plug in the commutator of Lie algebra to write

$$f_{bdc}\, f_{ade} + f_{abd}\, f_{cde} + f_{cad}\, f_{bde} = 0$$

this relations are obeyed by structure constants of any Lie group, e.g. $SU(n)$.

Define The generators in the adjoint representation by $(t^a)_{bc} = -i f_{abc}$ $\Rightarrow$ the above relation gives

$$[t^a, t^b] = i f_{abc}\, t^c$$

$\Rightarrow$ they obey the Lie algebra too!

Def. $D(\vec{A}) = e^{i A^a t^a}$ gives the <u>adjoint</u> representation of Lie group.

Tensor Method for SU(n)

Consider representation of SU(n) in terms of $n \times n$ unitary matrices U ($UU^+ = \mathbb{1}$) with $\det U = +1$.

Matrices U can be thought of as linear operators acting on the n-dim vectors $a_i = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in C_n$:

$$a_i \to a'_i = U_{ij} a_j .$$

Def.

A scalar product $a_i^* b_i = a \cdot b$ is invariant under su(n) transformations:

$$a_i^* b_i \to a'^*_i b'_i = U^*_{ij} a^*_j U_{ik} b_k = a^*_j \underbrace{U^+_{ji} U_{ik}}_{\delta_{jk}} b_k$$

$$= a^*_j b_j$$

Def.

Introduce upper indices: $a^i = a^*_i$, $U_i{}^j = U_{ij}$

$$U^i{}_j = U^*_{ij}$$

$$\Rightarrow \quad a_i \to a'_i = U_i{}^j a_j$$

$$a^i \to a'^i = U^i{}_j a^j$$

$\Rightarrow$ scalar product is $a^i b_i = a \cdot b$

unitarity $U_k{}^i U^k{}_j = U_{ki} U^*_{kj} = U_{ki} U^+_{jk} = \delta_{ij} \equiv \delta^i_j$

Def. a^i's form a basis for fundamental (defining) representation of $SU(n)$, denoted $\mathbf{n}$

a_i's form a basis for conjugate representation $\bar{\mathbf{n}}$

$\Rightarrow$ can construct any tensor $a^{i_1 \dots i_p}_{j_1 \dots j_q}$

$$a'^{\,i_1 \dots i_p}_{\;j_1 \dots j_q} = u^{i_1}{}_{k_1} \dots u^{i_p}{}_{k_p} \; u_{j_1}{}^{l_1} \dots u_{j_q}{}^{l_q} \; a^{k_1 \dots k_p}_{l_1 \dots l_q}$$

e.g. δ_i^j is invariant, so is Levi-Civita symbol $\varepsilon_{i_1 \dots i_n}$.

$\Rightarrow$ in general tensors form reducible representation of $SU(n)$.

$\Rightarrow$ to reduce them to irreducible representations note that permutation operator commutes with all u's: $\quad P_{12}\, a^{ij} = a^{ji} \quad \Rightarrow$

$$\Rightarrow P_{12}\, a'^{ij} = P_{12}\, u^i{}_k\, u^j{}_l\, a^{kl} = u^j{}_k\, u^i{}_l\, a^{kl} =$$

$$= (k \leftrightarrow l) = u^j{}_l\, u^i{}_k\, a^{lk} = u^i{}_k\, u^j{}_l\, P_{12}\, a^{kl}$$

$\Rightarrow$ organize all tensors by eigenstates of P_{12}: they could by <u>symmetric</u> & <u>anti-symmetric</u>.

a^{ij} : $S^{ij} = \frac{1}{2}(a^{ij} + a^{ji})$, $A^{ij} = \frac{1}{2}(a^{ij} - a^{ji})$ (51)

$\Rightarrow P_{12} S^{ij} = S^{ij}$; $P_{12} A^{ij} = -A^{ij}$

What is this good for?

Take a product of two representations:

$$a^i b^j = \frac{1}{2}(a^i b^j + a^j b^i) + \frac{1}{2}(a^i b^j - a^j b^i)$$

take SU(3) for example: a^i is **3**, $a^i b^j$ is $3 \otimes 3$.

$\frac{1}{2}(a^i b^j + a^j b^i)$ has 6 indep. components $\Rightarrow$ ~~transform~~ makes a basis for representation **6**.

$\frac{1}{2}(a^i b^j - a^j b^i)$ has 3 indep. components

$$= \frac{1}{2}\varepsilon^{ijk} \underbrace{\varepsilon_{klm} a^l b^m}_{c_k} \Rightarrow \text{it is } \bar{3}$$

$\Rightarrow$ we showed that $\boxed{3 \otimes 3 = 6 \oplus \bar{3}}$.

$$a^i b_j = \underbrace{\left(a^i b_j - \frac{1}{3}\delta^i_j a^k b_k\right)}_{\text{traceless } 3\times3 \text{ matrix}} + \underbrace{\frac{1}{3}\delta^i_j a^k b_k}_{1 \text{ (a singlet)}}$$

$\Rightarrow$ 8 d.of.freedom $\Rightarrow$ an 8 (adjoint representation)

$\Rightarrow \boxed{3 \otimes \bar{3} = 8 \oplus 1}$.