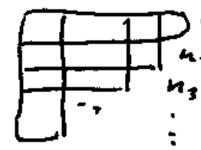
Last time: talked about Young Tableaux: $SU(N)$

$a^i \to \square$ representation N (fundamental), $a_{i_1} = \varepsilon_{i_1 \dots i_N} b^{i_2 \dots i_N} \Rightarrow$  } $N-1$ boxes

$\Rightarrow$ representation $\overline{N}$.

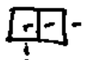
adjoint: $N-1$ {  $= N^2 - 1$.

In general  $n_1, n_2, n_3, \vdots$

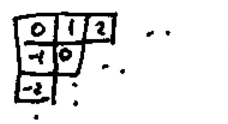
$n_1 \geq n_2 \geq n_3 \dots$

Dimension of representation: $$\boxed{d = \prod_{\text{boxes } i} \frac{N + D_i}{h_i}}$$

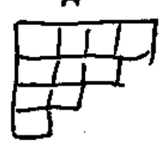
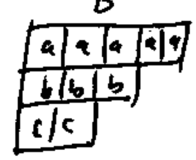
$h_i \sim$ hook length

 $h_i = 2$

$D_i \sim$ distance to 1st box (+1 for steps right, −1 for steps down)



Products of Young Tableaux:

A  $\otimes$ B 

(i) put boxes a on tableaux A building right & down

(ii) ibid for b's, c', ...

(iii) # a's $\geq$ # b's $\geq$ # c's reading from right to left from top row down, starting from $\forall$ spot/box.

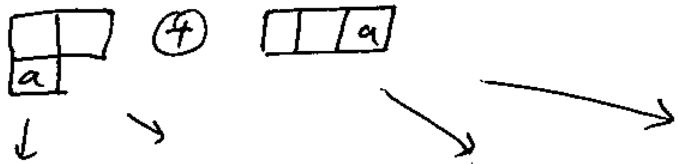
Using Young tableaux we showed that for $SU(3)$:

$$\boxed{3 \otimes \overline{3} = 1 \oplus 8} \qquad \boxed{3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10}$$

Example | $SU(2) \Rightarrow$ adjoint representation is

$3 =$  $\Rightarrow 3 \otimes 3 =$  $\Rightarrow$ calculate:

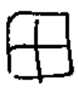
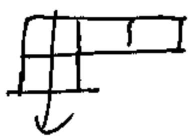
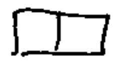
first step:

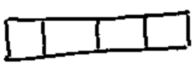


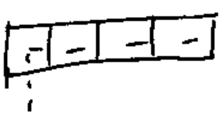
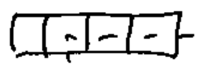
second step:

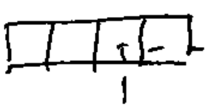
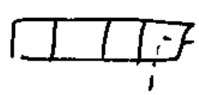
identical $\Rightarrow$ drop one

the column can be at most $N-1$ boxes long $\Rightarrow$

 $= \mathbb{1}$ singlet. ,  $=$  $= 3$

 $\Rightarrow$ find dimension:

 $h_1 = 4,\ D_1 = 0$;  $h_2 = 3,\ D_2 = 1$;

 $h_3 = 2,\ D_2 = 2$;  $h_4 = 1,\ D_4 = 3$.

$$\Rightarrow d = \prod_{i=1}^{4} \frac{2 + D_i}{h_i} = \frac{2}{4} \cdot \frac{3}{3} \cdot \frac{4}{2} \cdot \frac{5}{1} = 5 .$$

$$\Rightarrow \boxed{3 \otimes 3 = \mathbb{1} \oplus 3 \oplus 5} \quad \text{in } SU(2).$$

Quark Symmetries Revisited.

⇒ Isospin symmetry: we had an isospin operator $\vec{I}$ which was like angular momentum operator in 3d isospin fictitious space; as angular momentum it satisfied:

$$[I_a, I_b] = i\,\varepsilon_{abc}\, I_c$$

Compare with SU(2): group elements were $e^{i\vec{\alpha}\cdot\vec{J}}$ with $\vec{J} = \frac{1}{2}\vec{\sigma}$. We had the Lie algebra for generators:

$$[J_i, J_j] = i\,\varepsilon_{ijk}\, J_k .$$

⇒ isospin symmetry is an <u>SU(2) symmetry</u>!

Fundamental representation $\square$ of SU(2) is 2

⇒ a doublet ⇒ we saw a lot of isospin doublets

$$\begin{pmatrix} p \\ n \end{pmatrix}, \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \begin{pmatrix} \bar{K}^0 \\ K^- \end{pmatrix}, \ldots$$

$$2 \otimes \bar{2} = \square \otimes \square = \underbrace{\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}}_{\substack{\| \\ \mathbf{1}\ (\text{singlet})}} \oplus \underbrace{\begin{array}{|c|c|}\hline & \\ \hline \end{array}}_{\substack{\| \\ 3\ (\text{triplet})}} = 1 \oplus 3$$

⇒ can have isospin 3 = triplets: $\begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}, \begin{pmatrix} \rho^+ \\ \rho^0 \\ \rho^- \end{pmatrix}, \begin{pmatrix} \Sigma^+ \\ \Sigma^0 \\ \Sigma^- \end{pmatrix}, \ldots$

$\Rightarrow$ can also have iso-singlets: $\eta, \omega, \phi, \Lambda, \ldots$

Is this a symmetry of the Lagrangian that we wrote? Look at 2 flavors:

$$q(x) = \begin{pmatrix} u(x) \\ d(x) \end{pmatrix} \; ; \quad m = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \Rightarrow$$

$$\Rightarrow \mathcal{L}^{N_f=2}_{quarks} = \bar{q}\,(i\gamma\cdot\partial - m)\,q$$

First put $m=0 \Rightarrow \mathcal{L}_0 = \bar{q}^f\, i\gamma\cdot\partial\, q^f$

$\Rightarrow SU(2)$ flavor transformation would be

$$e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}} \Rightarrow \quad q \to q' = e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}} q \;, \quad \vec{\alpha} \sim \text{const} \; (x\text{-indep.})$$

$\Rightarrow \bar{q} \to \bar{q}' = \bar{q}\, e^{-i\vec{\alpha}\frac{\vec{\sigma}}{2}} \quad \Rightarrow \mathcal{L}_0$ is invariant:

$$\bar{q}\, i\gamma\cdot\partial q \to \bar{q}'\, i\gamma\cdot\partial q' = \bar{q}\, e^{-i\vec{\alpha}\frac{\vec{\sigma}}{2}}\, i\gamma\cdot\partial\, e^{i\vec{\alpha}\frac{\vec{\sigma}}{2}} q =$$

$$= \bar{q}\, i\gamma\cdot\partial q.$$

What about the mass term?

$$\bar{q} m q \to \bar{q}' m q' = \bar{q}\, e^{-i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}\, m\, e^{i\vec{\alpha}\frac{\vec{\sigma}}{2}} q$$

Write $m = \begin{pmatrix} \frac{m_u+m_d}{2} & 0 \\ 0 & \frac{m_u+m_d}{2} \end{pmatrix} + \begin{pmatrix} \frac{m_u-m_d}{2} & 0 \\ 0 & -\frac{m_u-m_d}{2} \end{pmatrix} \Rightarrow$

$$\Rightarrow \quad m = \frac{m_u + m_d}{2}\,\mathbb{1} + \frac{m_u - m_d}{2}\,\sigma^3$$

$$\Rightarrow \quad \bar{q}'\, m\, q' = \bar{q}\,\frac{m_u+m_d}{2}\,q + \frac{m_u-m_d}{2}\,\bar{q}\,\underbrace{e^{-i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}\,\sigma^3\, e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}}_{\neq\ \sigma^3}\,q$$

$\Rightarrow$ if $m_u = m_d$ $\Rightarrow$ get <u>exact</u> $SU(2)$ flavor symmetry (<u>global</u> $SU(2)$ symmetry ~ $\vec{\alpha}$ is independent of x^μ!)

as $m_u \neq m_d$ by a little bit $\Rightarrow$ $SU(2)$ flavor is (slightly) broken. ($\Rightarrow$ hadron masses are different)

$\Rightarrow$ in reality the symmetry group is much larger!

~ $SU(2)_R \times SU(2)_L$ ~ more on this later

(for massless quarks)

$\Rightarrow$ Now, put the strange quark back in:

$$q = \begin{pmatrix} u(x) \\ d(x) \\ s(x) \end{pmatrix}, \quad m = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

$$\Rightarrow \quad \mathcal{L}^{N_f=3}_{\text{quarks}} = \bar{q}\,(i\gamma\cdot\partial - m)\,q \quad \text{again.}$$

$\Rightarrow$ one can check that if $m_u = m_d = m_s$ then $\mathcal{L}$ is invariant under $SU(3)$ flavor transform:

$$q \to q' = e^{i\vec{\alpha}\cdot\vec{T}}\,q, \quad T^a = \frac{1}{2}\lambda^a, \quad \lambda^a \sim \text{Gell-Mann matrices}$$

$a = 1, 2, \dots, 8.$

$\Rightarrow$ as $m_u \neq m_d \neq m_s$, $SU(3)$ is not an exact flavor symmetry.

Now, let's look at mesons: $\bar{q} q$ ~ states

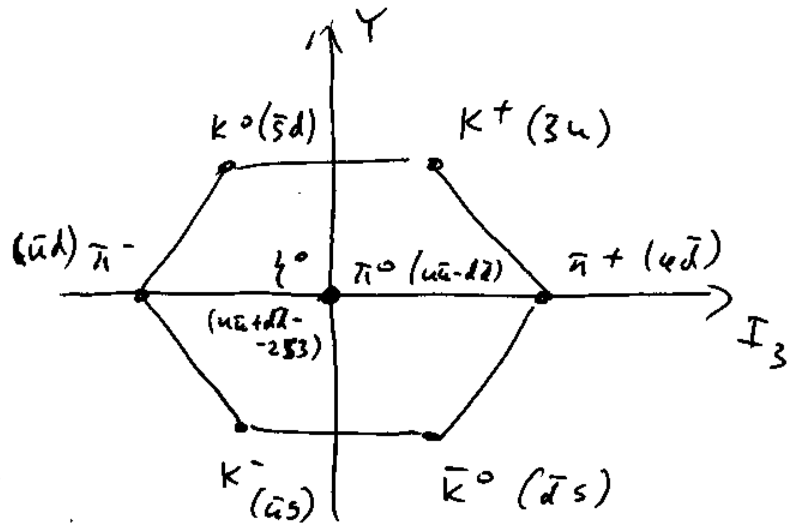
$\Rightarrow 3 \otimes \bar{3} = 1 \oplus 8 \Rightarrow$ there should be a flavor octet and singlet:

pseudoscalar mesons $\pi^+, \pi^-, \pi^0, \eta^0, K^+, K^0, \bar{K}^0, K^-$

form flavor-octet!
"The Eightfold Way"

η' ~ flavor singlet!
$\sim (\bar{u}u + \bar{d}d + \bar{s}s)\frac{1}{\sqrt{3}}$.

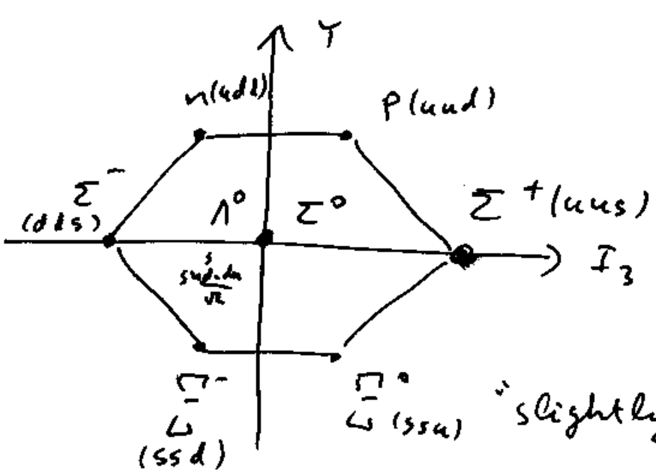
vector mesons ~ the same story!



What about baryons? qqq-states $\Rightarrow$

$3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$

$\frac{1}{2}^+$ baryons: $p, n, \Sigma^+, \Sigma^-, \Sigma^0, \Lambda^0, \Xi^0, \Xi^-$ ~ form an octet!



baryon decuplet ~ that's the 10!

$\Rightarrow$ as $SU(3)$ flavor is not exact, all masses are "slightly" different ~ broken symmetry!

$(m_{\Xi^0} = 1315\ \text{MeV},\ m_p = 938\ \text{MeV})$

Gell-Mann – Okubo Mass Formula

$\Rightarrow$ Note that $m_p \neq 2m_u + m_d$ $\Rightarrow$ most of the mass is due to gluonic interactions $\Rightarrow$

$\Rightarrow$ write

$$m_p = m_0 + 2m_u + m_d \simeq m_0 + 3m_u \quad (m_d \simeq m_u)$$

$$m_\Sigma = m_0 + 2m_u + m_s$$

$$m_\Xi = m_0 + m_u + 2m_s$$

$$m_\Lambda = m_0 + 2m_u + m_s$$

$m_\Lambda = m_\Sigma$

$\Rightarrow$ $$\frac{m_\Sigma + 3m_\Lambda}{2} = m_p + m_\Xi$$ for $\frac{1}{2}^+$ baryon octet.

$m_p = 938\,\text{MeV}$, $m_\Lambda = 1116\,\text{MeV}$, $m_\Xi = 1315\,\text{MeV}$, $m_\Sigma = 1189\,\text{MeV}$

LHS $= 2268.5$ MeV, RHS $= 2253$ MeV ~ close enough!

For $\frac{3}{2}^+$ baryon decuplet get

$$m_\Omega - m_{\Xi^*} = m_{\Xi^*} - m_{\Sigma^*} = m_{\Sigma^*} - m_{\Delta^*}$$

~ also works

~ was used to <u>predict</u> the mass of Ω-baryon.

Flavor SU(2) and SU(3) Symmetries.

Let's go back to 2-flavor QCD:

$$\mathcal{L}^{N_f=0}_{quarks} = \bar{q}\,(i\gamma\cdot\partial - m)\,q \quad , \quad m = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$$

We saw that if $m_u = m_d$ we have SU(2) flavor symmetry in the lagrangian.

$\Rightarrow$ However, masses of hadrons are much larger than current quark masses ($m_p \gg 2m_u + m_d$).

$\Rightarrow$ the flavor symmetry is more due to the fact that quark masses are small!

$\Rightarrow$ put $m_u = m_d = 0$

$$\Rightarrow \quad \mathcal{L} = \bar{q}\; i\gamma\cdot\partial\, q \,.$$

Write $$q = q_L + q_R = \underbrace{\frac{1-\gamma_5}{2}\,q}_{q_L} + \underbrace{\frac{1+\gamma_5}{2}\,q}_{q_R}$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \quad , \quad \{\gamma_5, \gamma^\mu\} = 0 \quad , \quad \gamma_5^{\dagger} = \gamma_5$$

$$\gamma_5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad , \quad \gamma_5\gamma_5 = \mathbb{1}$$

Projection operators $P_L = \frac{\mathbb{1}-\gamma_5}{2}$, $P_R = \frac{\mathbb{1}+\gamma_5}{2}$

$$\Rightarrow P_L^2 = \left(\frac{\mathbb{1}-\gamma_5}{2}\right)^2 = \frac{\mathbb{1} - 2\gamma_5 + \overbrace{\gamma_5^2}^{=\mathbb{1}}}{4} = P_L$$

$P_R^2 = P_R$, $\quad P_R P_L = \frac{1+\gamma_5}{2}\,\frac{1-\gamma_5}{2} = \frac{1-\gamma_5^2}{4} = 0$.

For massless particles they project on different helicity states. $P_L = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $\quad P_R = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Now, $\bar{q} = q^+\gamma^0 \Rightarrow \quad q^+ = q^+\left(\frac{1-\gamma_5}{2}\right) + q^+\left(\frac{1+\gamma_5}{2}\right) = q_L^+ + q_R^+$

$$\Rightarrow \bar{q} = \underbrace{\bar{q}\,\frac{1+\gamma_5}{2}}_{\bar{q}_L} + \underbrace{\bar{q}\,\frac{1-\gamma_5}{2}}_{\bar{q}_R} \qquad \text{as } \{\gamma_5, \gamma_0\} = 0 .$$

$$\Rightarrow \mathcal{L} = \left[\bar{q}\,\frac{1+\gamma_5}{2} + \bar{q}\,\frac{1-\gamma_5}{2}\right] i\gamma\cdot\partial \left[\frac{1-\gamma_5}{2} q + \frac{1+\gamma_5}{2} q\right]$$

survives (first with last term; second with first term)

$$\Rightarrow \boxed{\mathcal{L} = \bar{q}_L\, i\gamma\cdot\partial\, q_L + \bar{q}_R\, i\gamma\cdot\partial\, q_R}$$

Now, this lagrangian is <u>separately</u> invariant under $q_L \to e^{i\vec{\alpha}_L\cdot\frac{\vec{\sigma}}{2}} q_L$ and $q_R \to e^{i\vec{\alpha}_R\cdot\frac{\vec{\sigma}}{2}} q_R$

$\Rightarrow$ the net symmetry is $\boxed{SU(2)_L \otimes SU(2)_R}$. chiral symmetry

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$\Rightarrow$ Now add back the mass term with $m_u = m_d$:

$$-m\bar{q}q = -m\left[\bar{q}\,\frac{1+\gamma_5}{2} + \bar{q}\,\frac{1-\gamma_5}{2}\right]\left[\frac{1-\gamma_5}{2} q + \frac{1+\gamma_5}{2} q\right]$$

$= -m\left[\bar{q}_L q_R + \bar{q}_R q_L\right] \Rightarrow$ mixing $\Rightarrow$ need $\vec{\alpha}_R = \vec{\alpha}_L$

$\Rightarrow$ $SU(2)_L \otimes SU(2)_R$ is broken down to $SU(2)$.

What are the conserved currents of $SU(2)_R \otimes SU(2)_L$?

Noether theorem: every symmetry gives a conservation law!

Go back to the massless case:

$$\mathcal{L} = \bar{q}_L i\gamma\cdot\partial q_L + \bar{q}_R i\gamma\cdot\partial q_R$$

$$q_L \underset{SU(2)}{\rightarrow} e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}} q_L \Rightarrow \text{if } \vec{\alpha} \text{ is small } q_L \rightarrow \left(1 + i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}\right) q_L$$

$$= q_L + \delta q_L$$

$\Rightarrow$ $\delta\mathcal{L} = 0$ as it is a symmetry $\Rightarrow$

$$0 = \delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta q_L}\delta q_L + \frac{\delta\mathcal{L}}{\delta\bar{q}_L}\delta\bar{q}_L + \frac{\delta\mathcal{L}}{\delta(\partial_\mu q_L)}\delta(\partial_\mu q_L) +$$

$$+ \delta(\partial_\mu\bar{q}_L)\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{q}_L)} = \underbrace{\left[\frac{\delta\mathcal{L}}{\delta q_L} - \partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu q_L)}\right]}_{=0\ (\text{EOM})}\delta q_L + \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu q_L)}\delta q_L\right)$$

$$+ \delta\bar{q}_L\underbrace{\left[\frac{\delta\mathcal{L}}{\delta\bar{q}_L} - \partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{q}_L)}\right]}_{=0\ (\text{EOM})} + \partial_\mu\left[\delta\bar{q}_L\cancel{\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{q}_L)}}\right]$$

$\rightarrow 0$ for our $\mathcal{L}$

$$\Rightarrow 0 = \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu q_L)}\delta q_L\right)$$

$$\Rightarrow 0 = \partial_\mu\left[\bar{q}_L i\gamma^\mu\delta q_L\right] = \partial_\mu\left[\bar{q}_L i\gamma^\mu i\frac{\vec{\alpha}\cdot\vec{\sigma}}{2}q_L\right]$$