Classical Yang-Mills Equations (see problem 1 in HW4)

start with pure YM Lagrangian: $\mathcal{L}_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$

with $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$.

Let's find the EOM: $\boxed{\frac{\delta \mathcal{L}}{\delta A^a_\mu} - \partial_\nu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\nu A^a_\mu)} \right) = 0}$

$$\frac{\delta \mathcal{L}}{\delta(\partial_\nu A^a_\mu)} = -\frac{1}{4} \frac{\delta(F^b_{\alpha\beta} F^{b\alpha\beta})}{\delta(\partial_\nu A^a_\mu)} = -\frac{1}{2} F^{b\alpha\beta} \frac{\delta F^b_{\alpha\beta}}{\delta(\partial_\nu A^a_\mu)} =$$

$$= -F^{a\nu\mu} = F^{a\mu\nu} \quad \text{(just like for photon field)}$$

$$\frac{\delta \mathcal{L}}{\delta A^a_\mu} = -\frac{1}{4} \frac{\delta(F^b_{\alpha\beta} F^{b\alpha\beta})}{\delta A^a_\mu} = -\frac{1}{2} F^{b\alpha\beta} \frac{\delta F^b_{\alpha\beta}}{\delta A^a_\mu} =$$

$$= -\frac{1}{2} F^{b\alpha\beta} \cdot \frac{\delta(g f^{bcd} A^c_\alpha A^d_\beta)}{\delta A^a_\mu} = -\frac{1}{2} F^{b\alpha\beta} g f^{bcd} \cdot$$

$$\left[\delta^\mu_\alpha \delta^{ac} A_\beta{}^d + A_\alpha{}^c \delta_\beta{}^\mu \delta^{ad} \right] = -\frac{1}{2} g \left[f^{bad} A^d_\beta F^{b\mu\beta} + \right.$$

$$\left. + f^{bca} A^c_\alpha F^{b\alpha\mu} \right] = -g f^{adb} A^d_\nu F^{b\mu\nu} = -g f^{abc} A^b_\nu F^{c\mu\nu}$$

$$\Rightarrow \text{EOM are: } -g f^{abc} A_\nu{}^b F^{c\mu\nu} - \partial_\nu F^{a\mu\nu} = 0$$

$$\Rightarrow \partial_\nu F^{a\nu\mu} = -g f^{abc} A^b_\nu F^{c\nu\mu} \equiv J^\mu$$

(solution of problem 1 in HW4)

rewrite EOM as

$$\left[\delta^{ac}\,\partial_\nu + g f^{abc} A_\nu^{b}\right] F^{c\nu\mu} = 0$$

looks like a covariant derivative ...

take EOM from previous page & multiply by T^a & sum over a's

$\Rightarrow$ as $F_{\mu\nu} = \sum_{a=1}^{N^2-1} T^a F^a_{\mu\nu}$ $\Rightarrow$

$$\Rightarrow \partial_\nu F^{\nu\mu} = -g \underbrace{f^{abc} T^a}_{-i[T^b, T^c]} A^b_\nu F^{c\nu\mu} = ig\left[A_\nu, F^{\nu\mu}\right]$$

$$\Rightarrow \partial_\nu F^{\nu\mu} - ig\left[A_\nu, F^{\nu\mu}\right] = 0$$ is the EOM.

Def. define a covariant derivative acting on adjoint fields: $\boxed{\mathcal{D}_\mu = \partial_\mu - ig\left[A_\mu(x), \,\bullet\,\right]}$

$\Rightarrow$ Yang-Mills equations now read:

$$\boxed{\mathcal{D}_\mu F^{\mu\nu} = 0}$$

(just like Maxwell eqn's in vacuum).