Homework Set No. 1, Physics 880.08

Deadline – Wednesday, October 14, 2009

1. Consider a real scalar field theory with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\,\phi^2 - \frac{\lambda}{4!}\,\phi^4.$$

(a) (5 pts) Construct Euler-Lagrange equation for this theory.

(b) (5 pts) Find the energy-momentum tensor $T^{\mu\nu}$ for this theory and show explicitly that it is conserved, $\partial_\mu T^{\mu\nu} = 0$, for the fields satisfying Euler-Lagrange equation found in part (a).

2. (10 pts) The Lagrangian density for a two-component real scalar field $\vec{\phi} = (\phi_1, \phi_2)$ is given by

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - \frac{m^2}{2}\,|\vec{\phi}|^2.$$

This Lagrangian is invariant under rotations by "angle" α in the field component space:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{1}$$

Find the conserved current j^μ and charge Q corresponding to this symmetry. (Hint: it may be easier to use the totally antisymmetric 2d Levi-Civita symbol ϵ^{ij} with $\epsilon^{12} = -\epsilon^{21} = 1$, $\epsilon^{11} = \epsilon^{22} = 0$ in arriving at the answer.)

3. Consider generators of some Lie group obeying Lie algebra commutation relations

$$[X_a, X_b] = i\,f_{abc}\,X_c \tag{2}$$

with anti-symmetric structure constants f_{abc}.

(a) (5 pts) Prove the Jacobi identity

$$[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0$$

by expanding out the commutators.

(b) (5 pts) Use the commutation relation (2) for X_a's in the Jacobi identity to show that

$$f_{bcd}\,f_{ade} + f_{abd}\,f_{cde} + f_{cad}\,f_{bde} = 0,$$

which is also often referred to as the Jacobi identity.

4. (10 pts) Using Gell-Mann matrices (and their commutators) find the structure constants f^{156} and f^{678} of the group $SU(3)$.

5. (10 pts) In class we defined the generators of Lorentz group by

$$L_{\mu\nu} = i\,(x_\mu\,\partial_\nu - x_\nu\,\partial_\mu).$$

Show that these generators obey the following algebra

$$[L_{\mu\nu}, L_{\rho\sigma}] = i\,\eta_{\nu\rho}\,L_{\mu\sigma} - i\,\eta_{\mu\rho}\,L_{\nu\sigma} - i\,\eta_{\nu\sigma}\,L_{\mu\rho} + i\,\eta_{\mu\sigma}\,L_{\nu\rho}.$$