Homework Set No. 2, Physics 880.08

Deadline – Monday, November 2, 2009

1. (10 pts) In class we showed that Dirac spinors transform as

$$\psi_D(x) \, \rightarrow \, \psi'_D(x') \, = \, \left(\begin{array}{cc} e^{\frac{i}{2}\,\vec{\sigma}\cdot(\vec{\theta}-i\vec{\xi})} & 0 \\ 0 & e^{\frac{i}{2}\,\vec{\sigma}\cdot(\vec{\theta}+i\vec{\xi})} \end{array} \right) \psi_D(x) \tag{1}$$

under Lorentz transformations. Show that this transformation rule is equivalent to

$$\psi_D(x) \, \rightarrow \, \psi'_D(x') \, = \, e^{-\frac{i}{4}\,\omega^{\mu\,\nu}\,\sigma_{\mu\,\nu}}\,\psi_D(x)$$

where

$$\sigma_{\mu\,\nu} \, = \, \frac{i}{2}\,[\gamma_\mu, \gamma_\nu].$$

As usual $\xi^i = \omega^{0i}$ and $\theta_i = \frac{1}{2}\,\epsilon_{ijk}\,\omega_{jk}$. You can consider boosts and rotations separately for the full credit.

2. **(a)** (5 pts) Complete the proof started in class that

$$\bar{\psi}\,\gamma^\mu\,\psi$$

is a 4-vector by showing that it transforms like one under infinitesimal boosts. ψ is the Dirac spinor which transforms according to Eq. (1).

(b) (5 pts) Prove that

$$\bar{\psi}\,\gamma^\mu\,\gamma^5\,\psi$$

is a 4-vector under both boosts and rotations. What happens to it under parity?

3. (10 pts) Writing Dirac spinor in terms of two-component Weyl spinors

$$\psi_D(x) \, = \, \left(\begin{array}{c} \chi_L \\ \chi_R \end{array} \right)$$

one can see from Eq. (1) above that Weyl spinors transform under Lorentz transformations as

$$\chi_L(x) \, \rightarrow \chi'_L(x') \, = \, e^{\frac{i}{2}\,\vec{\sigma}\cdot(\vec{\theta}-i\vec{\xi})}\,\chi_L(x)$$
$$\chi_R(x) \, \rightarrow \chi'_L(x') \, = \, e^{\frac{i}{2}\,\vec{\sigma}\cdot(\vec{\theta}+i\vec{\xi})}\,\chi_R(x).$$

Show that $\sigma^2\,\chi_L^*$ transforms as a right-handed Weyl spinor. Here $*$ denotes complex conjugation and σ^2 is a Pauli matrix.

4. Consider a massive Dirac field ψ with mass m.

a. (2 pts) Starting with Dirac equation

$$[i\gamma^\mu \partial_\mu - m]\psi = 0$$

derive an equation for $\bar{\psi}$.

b. (3 pts) Using the result of part **a** show that the electromagnetic current

$$j_\mu = \bar{\psi}\gamma_\mu\psi$$

is conserved at the classical level, i.e., show that $\partial_\mu j^\mu = 0$.

c. (5 pts) Defining $\gamma^5 = i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3$ use the anti-commutation relations for γ-matrices $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ to show that

$$\left\{\gamma^\mu, \gamma^5\right\} = 0.$$

Use this result to show that the divergence of the axial vector current

$$j^{5\,\mu} = \bar{\psi}\gamma^\mu\gamma^5\psi$$

is

$$\partial_\mu\, j^{5\,\mu} \;=\; 2\,i\,m\,\bar{\psi}\,\gamma^5\,\psi. \tag{2}$$

That is the axial current is conserved for massless particles (in this classical theory).