Homework Set No. 3, Physics 880.08

Deadline – Wednesday, November 18, 2009

1. Consider a free (real) scalar theory with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2}\varphi^2.$$

Define the Hamiltonian by

$$H(t) = \int d^3x \left[\pi(\vec{x},t)\,\dot{\varphi}(\vec{x},t) - \mathcal{L}\right].$$

a. (3 pts) Show that for classical field configurations

$$\frac{d}{dt}H(t) = 0.$$

b. (2 pts) Write $H(t)$ in terms of π and φ with no $\dot{\varphi}$'s appearing.

c. (5 pts) Now imagine that the field is quantized. Assuming that the classical Klein-Gordon equation holds for the operator $\varphi(\vec{x},t)$ use canonical quantization commutators

$$[\varphi(\vec{x},t), \pi(\vec{x}',t)] = i\delta(\vec{x}-\vec{x}')$$

(with all other equal-time commutators being zero) to show that $H(t)$ (now an operator) generates time translations, i.e., show that

$$\begin{aligned} i\,\partial_0 \varphi &= [\varphi, H(t)] \\ i\,\partial_0 \pi &= [\pi, H(t)]\,. \end{aligned}$$

2. The same as in problem 1, but now for Dirac field: start with a theory with Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi.$$

a. (3 pts) Construct a Hamiltonian $H(t)$ and show that for classical field configurations

$$\frac{d}{dt}H(t) = 0.$$

b. (2 pts) Write $H(t)$ in terms of π and ψ.

c. (5 pts) For quantized field ψ use the anti-commutation relations

$$\left\{\psi_\alpha(\vec{x},t), \psi^\dagger_\beta(\vec{x}',t)\right\} = \delta_{\alpha\beta}\,\delta(\vec{x}-\vec{x}')$$

along with Dirac equation to show that

$$\begin{aligned} i\,\partial_0 \psi_\alpha &= [\psi_\alpha, H(t)] \\ i\,\partial_0 \bar{\psi}_\alpha &= \left[\bar{\psi}_\alpha, H(t)\right]. \end{aligned}$$

3. In class we quantized free real scalar field theory of problem 1. The field operator was shown to be

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\, 2\, E_k} \left[\hat{a}_{\vec{k}}\, e^{-i\,k\cdot x} + \hat{a}^\dagger_{\vec{k}}\, e^{i\,k\cdot x}\right] \tag{1}$$

with the particle creation and annihilation operators obeying the following commutation relations

$$\left[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}\right] = (2\pi)^3\, 2\, E_k\, \delta^3(\vec{k} - \vec{k}'), \quad \left[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}\right] = \left[\hat{a}^\dagger_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}\right] = 0.$$

Above $k \cdot x = E_k\, t - \vec{k} \cdot \vec{x}$. The Hamiltonian was shown to be

$$\hat{H} = \int \frac{d^3k}{(2\pi)^3\, 2\, E_k}\, E_k\, \hat{a}^\dagger_{\vec{k}}\, \hat{a}_{\vec{k}}.$$

(a) (5 pts) Find the commutators $[\hat{H}, \hat{a}_{\vec{k}}]$ and $[\hat{H}, \hat{a}^\dagger_{\vec{k}}]$.

(b) (15 pts) Use the results of part (a) to write the particle creation and annihilation operators in Heisenberg representation defined by

$$\begin{aligned} \hat{a}^H_{\vec{k}}(t) &= e^{i\,\hat{H}\,t}\, \hat{a}_{\vec{k}}\, e^{-i\,\hat{H}\,t} \\ \hat{a}^{H\,\dagger}_{\vec{k}}(t) &= e^{i\,\hat{H}\,t}\, \hat{a}^\dagger_{\vec{k}}\, e^{-i\,\hat{H}\,t} \end{aligned} \tag{2}$$

in terms of E_k, $\hat{a}_{\vec{k}}$ and $\hat{a}^\dagger_{\vec{k}}$ (i.e., eliminate $\hat{H}$ in Eqs. (2) above). Rewrite the field ϕ from Eq. (1) in terms of the obtained $\hat{a}^H_{\vec{k}}(t)$ and $\hat{a}^{H\,\dagger}_{\vec{k}}(t)$.

(Hint: you may find the following identity useful

$$\left[\hat{H}, \left[\hat{H}, \ldots, \left[\hat{H}, \hat{\mathcal{O}}\right] \ldots\right]\right] = \sum_{m=0}^{N} \frac{N!}{m!\,(N-m)!}\, \hat{H}^{N-m}\, \hat{\mathcal{O}}\, \left(-\hat{H}\right)^m$$

where $\hat{\mathcal{O}}$ is an arbitrary operator and there are N commutators on the left hand side. If you use this identity, you would have to prove it for full credit. The proof can be done by induction.)