Homework Set No. 3, Physics 880.08

Deadline – Friday, May 14, 2010

1. Show explicitly that the time evolution kernel we derived in class for a free non-relativistic particle

$$U_{free}(q,t;q',t') \, = \, \sqrt{\frac{m}{2\,\pi\,i\,\hbar\,(t-t')}} \, e^{\frac{i}{\hbar}\,\frac{m}{2}\,\frac{(q-q')^2}{t-t'}}$$

is unitary.

a (5 pts) First show that

$$\int\limits_{-\infty}^{\infty} dq'\, U_{free}(q,t;q',t')\, U_{free}(q',t';q'',t'') \, = \, U_{free}(q,t;q'',t'').$$

b (5 pts) Then demonstrate that

$$\lim_{t \to t'} U_{free}(q,t;q',t') \, = \, \delta(q-q').$$

Explain why the results of parts **a** and **b** prove unitarity of the time evolution kernel.

2. a. (5 pts) By explicitly expanding the exponentials on the left-hand-side and carrying out the Grassmann integrals show that the following relation holds

$$\int d\bar{\chi}_1\, d\chi_1\, d\bar{\chi}_2\, d\chi_2 \, \exp\left[-\sum_{i,j} a_{ij}\,\bar{\chi}_i\,\chi_j\right] \exp\left[\sum_k (\bar{\chi}_k\,\xi_k + \bar{\xi}_k\,\chi_k)\right] \, = \, (\det A)\, \exp\left[\sum_{i,j} \bar{\xi}_i A^{-1}_{ij}\,\xi_j\right]$$

where χ_i and ξ_j are Grassmann variables, and A is a 2×2 matrix with elements a_{ij}.

b (5 pts) Show that

$$-i\frac{\partial}{\partial \bar{\xi}} F \, = \, \chi\, F \, = \, F\,\chi$$
$$i\frac{\partial}{\partial \xi} F \, = \, \bar{\chi}\, F \, = \, F\,\bar{\chi}$$

for the function

$$F \, = \, \exp\left[i\,(\bar{\xi}\,\chi + \bar{\chi}\,\xi)\right].$$

Here χ and ξ are Grassmann variables.

3. (10 pts) Consider a non-Abelian gauge theory with the gauge field A^a_μ and the Lagrangian

$$\mathcal{L} \,=\, -\frac{1}{4}\, F^a_{\mu\nu}\, F^{a\,\mu\nu}.$$

Here

$$F^a_{\mu\nu} \,=\, \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f^{abc}\, A^b_\mu\, A^c_\nu$$

with f^{abc} the structure constants of the gauge group $SU(N)$.

Write the equations of motion for this theory. If we define $J^{a\,\mu}$ by

$$\partial_\nu\, F^{a\,\nu\mu} \,=\, J^{a\,\mu}$$

what is $J^{a\,\mu}$ for the above Lagrangian?

4. (10 pts) When constructing gauge-invariant Lagrangian for the non-Abelian gauge field A^a_μ one may consider another Lorentz-invariant

$$I \,=\, \epsilon^{\mu\nu\rho\sigma}\, F^a_{\mu\nu}\, F^a_{\rho\sigma}.$$

Show that this term can be written as a 4-divergence,

$$I \,=\, \partial_\mu\, K^\mu$$

and find the 4-vector K^μ explicitly in terms of the field A^a_μ. Why can not the invariant I serve as the Lagrangian for the non-Abelian field?

(Hint: you may find the identity

$$f^{abe}\, f^{cde} \,=\, \frac{2}{N}\, (\delta^{ac}\, \delta^{bd} - \delta^{ad}\, \delta^{bc}) + d^{ace}\, d^{bde} - d^{bce}\, d^{ade}$$

useful. Here the gauge group is $SU(N)$ and d^{abc} is the absolutely symmetric object defined by

$$d^{abc} \,=\, 2\,\mathrm{Tr}\left(T^a\, \{T^b, T^c\}\right)$$

with T^a the generators of $SU(N)$ in the fundamental representation.)