Homework Set No. 4, Physics 880.08

Deadline – Friday, December 4, 2009

1. Time-ordered product of real scalar fields is defined by

$$\mathrm{T}\phi(x)\,\phi(y) \;=\; \theta(x^0 - y^0)\,\phi(x)\,\phi(y) + \theta(y^0 - x^0)\,\phi(y)\,\phi(x),$$

where ϕ's are operators in Heisenberg picture.

a. (5 pts) In a free scalar field theory with mass m use Klein-Gordon equation along with the canonical commutation relations to show that

$$[\partial^2 + m^2]\,\mathrm{T}\phi(x)\,\phi(y) \;=\; -i\,\delta^4(x-y)$$

where the derivative squared (the D'Alembertian) is taken with respect to 4-coordinates x.

b. (10 pts) Similar to what we did in class for retarded Green function, find an explicit expression for the Feynman propagator in coordinate space in a massless ($m = 0$) theory by performing the following integral

$$D_F(x-y) \;=\; \int \frac{d^4k}{(2\,\pi)^4}\, e^{-i\,k\cdot(x-y)}\, \frac{i}{k^2 + i\,\epsilon}.$$

Is Feynman propagator causal?

2. Time-ordered product of Dirac spinors is defined by

$$\mathrm{T}\,\psi_\alpha(x)\,\bar\psi_\beta(y) \;=\; \theta(x^0 - y^0)\,\psi_\alpha(x)\,\bar\psi_\beta(y) - \theta(y^0 - x^0)\,\bar\psi_\beta(y)\,\psi_\alpha(x)$$

where $\alpha, \beta = 1, 2, 3, 4$ are Dirac indices. ψ and $\bar\psi$ are operators in Heisenberg picture.

a. (3 pts) In a free Dirac field theory with mass m use Dirac equation and canonical anti-commutation relations to show that (α' is summed over)

$$[i\,\gamma^\mu\,\partial_\mu - m]_{\alpha\,\alpha'}\,\mathrm{T}\,\psi_{\alpha'}(x)\,\bar\psi_\beta(y) \;=\; i\,\mathbf{1}_{\alpha\beta}\,\delta^4(x-y).$$

b. (10 pts) Perform explicit calculation to show that the Feynman propagator for fermions is

$$S_F(x-y) \;\equiv\; \langle 0|\mathrm{T}\,\psi(x)\,\bar\psi(y)|0\rangle \;=\; \int \frac{d^4k}{(2\,\pi)^4}\, e^{-i\,k\cdot(x-y)}\, \frac{i\,(\gamma\cdot k + m)}{k^2 - m^2 + i\,\epsilon}.$$

(We did much of this in class, but it is still useful to do by yourself.)

3. If a photon was a *massive* particle of mass m its Lagrangian density (in the absence of sources) would have been given by the so-called Proca Lagrangian

$$\mathcal{L}_{Proca} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu \tag{1}$$

where A_μ is the 4-vector photon field.

(a) (5 pts) Find the equations of motion for the Proca Lagrangian (known as the Proca equations).

(b) (5 pts) Take a 4-divergence of the Proca equations obtained in (a) to show that if $m \neq 0$ Proca equations require Lorenz gauge condition $\partial_\mu A^\mu = 0$ to always be valid. (Hence Proca Lagrangian in Eq. (1) is not gauge-invariant!) Rewrite Proca equations imposing Lorenz gauge condition.

4. (7 pts) Construct scalar electrodynamics following steps similar to what we did in class in deriving QED. First consider a complex scalar field theory with the Lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi.$$

It is invariant under a global $U(1)$ symmetry: $\phi(x) \to e^{i\alpha} \phi(x)$ with α a real number. Gauge this Lagrangian by modifying it to have a *local* $U(1)$ symmetry

$$\phi(x) \to e^{i\alpha(x)} \phi(x) \tag{2}$$

with the help of a new vector field A_μ which transforms as

$$A_\mu \to A_\mu - \frac{1}{e} \partial_\mu \alpha(x). \tag{3}$$

Finally add the Lagrangian density for the free vector field. What is the resulting Lagrangian? Show explicitly that it is invariant under the local $U(1)$ transformations given by Eqs. (2) and (3).