Homework Set No. 1, Physics 880.08

Deadline – Monday, October 18, 2010

1. Consider a real scalar interacting field theory with the Lagrangian density

$$\mathcal{L} \,=\, \frac{1}{2}\,\partial_\mu \phi\, \partial^\mu \phi - \frac{m^2}{2}\,\phi^2 - \frac{\lambda}{3!}\,\phi^3$$

where λ is a real number.

(a) (5 pts) Construct Euler-Lagrange equation for this theory.

(b) (5 pts) Find the energy-momentum tensor $T^{\mu\nu}$ for this theory and show explicitly that it is conserved, $\partial_\mu T^{\mu\nu} = 0$, for the fields satisfying Euler-Lagrange equation found in part (a).

2. The Lagrangian density for a two-component complex scalar field

$$\vec{\phi} = \left(\begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right)$$

is given by

$$\mathcal{L} \,=\, \partial_\mu \vec{\phi}^{\,\dagger} \cdot \partial^\mu \vec{\phi} - m^2\, \vec{\phi}^\dagger\, \vec{\phi}$$

where Hermitean conjugation is defined by

$$\vec{\phi}^{\,\dagger} \,=\, (\phi_1^*, \phi_2^*)\,.$$

(a) (3 pts) Show that the above Lagrangian is invariant under the following global $SU(2)$ symmetry

$$\phi_i \,\rightarrow\, \phi_i' \,=\, \left(\exp\left\{ i\, \frac{\vec{\alpha}\cdot\vec{\sigma}}{2} \right\} \right)_{ij}\, \phi_j$$

with $\vec{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$ an arbitrary coordinate-independent vector, $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ the Pauli matrices, and $i, j = 1, 2$. Summation over repeated indices is assumed.

(b) (7 pts) Find the conserved current j_μ^a and charge Q^a corresponding to this symmetry (here $a = 1, 2, 3$).

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3. Consider generators of some Lie group obeying Lie algebra commutation relations

$$[X_a, X_b] \,=\, i\, f_{abc}\, X_c \tag{1}$$

with anti-symmetric structure constants f_{abc}.

(a) (5 pts) Prove the Jacobi identity

$$[X_a,[X_b,X_c]] + [X_b,[X_c,X_a]] + [X_c,[X_a,X_b]] = 0$$

by expanding out the commutators.

(b) (5 pts) Use the commutation relation (1) for X_a's in the Jacobi identity to show that

$$f_{bcd}\, f_{ade} + f_{abd}\, f_{cde} + f_{cad}\, f_{bde} = 0,$$

which is also often referred to as the Jacobi identity.

4. (10 pts) Using Gell-Mann matrices (and their commutators) find the structure constants f^{147} and f^{458} of the group $SU(3)$.

5. (10 pts) In class we defined the generators of Lorentz group by

$$L_{\mu\nu} \,=\, i\,(x_\mu\, \partial_\nu - x_\nu\, \partial_\mu).$$

Show that these generators obey the following algebra

$$[L_{\mu\nu}, L_{\rho\sigma}] \,=\, i\,\eta_{\nu\rho}\, L_{\mu\sigma} - i\,\eta_{\mu\rho}\, L_{\nu\sigma} - i\,\eta_{\nu\sigma}\, L_{\mu\rho} + i\,\eta_{\mu\sigma}\, L_{\nu\rho}.$$