Homework Set No. 2, Physics 880.08
Deadline – Wednesday, November 3, 2010

1. (10 pts) In class we showed that Dirac spinors transform as

$$\psi_D(x) \;\to\; \psi'_D(x') \;=\; \begin{pmatrix} e^{-\frac{i}{2}\,\vec{\sigma}\cdot(\vec{\theta}+i\vec{\xi})} & 0 \\ 0 & e^{-\frac{i}{2}\,\vec{\sigma}\cdot(\vec{\theta}-i\vec{\xi})} \end{pmatrix} \psi_D(x) \tag{1}$$

under Lorentz transformations. Show that this transformation rule is equivalent to

$$\psi_D(x) \;\to\; \psi'_D(x') \;=\; e^{-\frac{i}{4}\,\omega^{\mu\,\nu}\,\sigma_{\mu\,\nu}}\,\psi_D(x)$$

where

$$\sigma_{\mu\,\nu} \;=\; \frac{i}{2}\,[\gamma_\mu, \gamma_\nu].$$

As usual $\xi^i = \omega^{0i}$ and $\theta_i = \frac{1}{2}\,\epsilon_{ijk}\,\omega_{jk}$. You may consider boosts and rotations separately for the full credit.

2. **(a)** (5 pts) Complete the proof started in class that

$$\bar{\psi}\,\gamma^\mu\,\psi$$

is a 4-vector by showing that it transforms like one under infinitesimal boosts. ψ is the Dirac spinor which transforms according to Eq. (1).

(b) (5 pts) Prove that

$$\bar{\psi}\,\gamma^\mu\,\gamma^5\,\psi$$

is a 4-vector under both boosts and rotations. What happens to it under parity?

3. Consider a real scalar field theory with the Lagrangian density

$$\mathcal{L} \;=\; \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{\lambda}{4}\,\left(\phi^2 - v^2\right)^2 \tag{2}$$

where λ and v are some positive real numbers. Imagine that the theory lives in $1+1$ space-time dimensions labeled (t, x).

(a) (3 pts) Construct the Hamiltonian for the theory. Show that, for time-independent fields $\phi(t, x) = \phi(x)$, the energy minima (the vacua) of the Hamiltonian are given by

$$\phi \;=\; \pm v.$$

(b) (7 pts) Find the time-independent solution of the equations of motion for the Lagrangian (2) that interpolates between the two vacua. That is find the solution $\phi(x)$ satisfying the following conditions

$$\begin{aligned} \phi(x=-\infty) &= -v \\ \phi(x=+\infty) &= v. \end{aligned}$$

(You may also require that $\phi(x=0)=0$ for simplicity.) Such solution is known as the 'kink' solution and is the simplest example of a soliton.

4. Consider a massive Dirac field ψ with mass m.

a. (2 pts) Starting with Dirac equation

$$[i\gamma^\mu \partial_\mu - m]\psi = 0$$

derive an equation for $\bar{\psi}$.

b. (3 pts) Using the result of part **a** show that the electromagnetic current

$$j_\mu = \bar{\psi}\gamma_\mu \psi$$

is conserved at the classical level, i.e., show that $\partial_\mu j^\mu = 0$.

c. (5 pts) Defining $\gamma^5 = i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3$ use the anti-commutation relations for γ-matrices $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ to show that

$$\{\gamma^\mu, \gamma^5\} = 0.$$

Use this result to show that the divergence of the axial vector current

$$j^{5\,\mu} = \bar{\psi}\gamma^\mu\gamma^5\psi$$

is

$$\partial_\mu\, j^{5\,\mu} \;=\; 2\,i\,m\,\bar{\psi}\,\gamma^5\,\psi.$$

That is the axial current is conserved for massless particles (in this classical theory).