Homework Set No. 4, Physics 880.08

Deadline – noon on Tuesday, March 15, 2011

1. Construct the momentum-space Feynman rules for the following theories (draw all the propagators and vertices, and find their contributions to the diagrams):

a. (10 pts) Scalar electrodynamics, with the Lagrangian density

$$\mathcal{L} = (D_\mu \varphi)^* D^\mu \varphi - m^2 \varphi^* \varphi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

where $D_\mu = \partial_\mu + i\,e\,A_\mu$. Here φ is a complex scalar field, A_μ is a real gauge field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

b. (5 pts) Complex scalar theory, with the Lagrangian density

$$\mathcal{L} = (\partial_\mu \varphi)^* \partial^\mu \varphi - \lambda^2 \varphi^* \varphi (\varphi^* \varphi - v^2)^2,$$

where λ and v are real positive constants and φ is a complex scalar field.

Note that while for the photon propagator you can simply use the expression given in class, I would like you to derive the propagator for the complex scalar field in part **a**.

2. Consider φ^4 scalar theory with the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi\, \partial^\mu \varphi - \frac{m^2}{2} \varphi^2 - \frac{\lambda}{4!} \varphi^4.$$

Below use Pauli-Villars regularization with two new particles, such that the propagators of the internal lines are replaced by

$$\frac{i}{k^2 - m^2 + i\,\epsilon} \to \frac{i}{k^2 - m^2 + i\,\epsilon} + \sum_{j=1}^{2} \frac{i\,C_j}{k^2 - M_j^2 + i\,\epsilon}$$

with $C_1 = 1$, $C_2 = -2$, and $M_1^2 = m^2 + 2\,M^2$, $M_2^2 = m^2 + M^2$. M is some UV cutoff.

a. (30 pts) Calculate regularized one-loop 1PI corrections to the propagator and to the four-point Green function, keeping only divergent and finite terms at large M (i.e., dropping all terms which vanish in the $M \to \infty$ limit).

b. (15 pts) Set up the renormalization program for the ϕ^4 theory at one loop: that is, find all counterterms and coefficients in front of them using "on-shell" renormalization conditions:

$$\begin{aligned} \Sigma(p^2 = m^2) &= 0, \\ \left.\frac{\partial \Sigma(p^2)}{\partial p^2}\right|_{p^2=m^2} &= 0, \\ \Gamma^4(s = 4\,m^2, t = 0, u = 0) &= -i\,\lambda. \end{aligned}$$

Here $-i\,\Sigma(p^2)$ is the sum of all 1PI corrections to the propagator, and $\Gamma^4(s,t,u)$ is the truncated 1PI 4-point Green function. Also m is the physical particle mass, and λ is the physical coupling.

In particular show that the renormalized 4-point function $\Gamma^4_R(s,t,u)$ is given by

$$\begin{aligned} \Gamma^4_R(s,t,u) \;=\; -i\Bigg\{\lambda + \frac{\lambda^2}{32\,\pi^2}\int_0^1 dx\,\Bigg[\ln\left(\frac{m^2 - s\,x\,(1-x)}{m^2 - 4\,m^2 x\,(1-x)}\right) + \ln\left(\frac{m^2 - t\,x\,(1-x)}{m^2}\right) \\ + \ln\left(\frac{m^2 - u\,x\,(1-x)}{m^2}\right)\Bigg] + o(\lambda^3)\Bigg\}. \end{aligned}$$

For a 4-point function with two external lines having incoming 4-momenta p_1^μ and p_2^μ and two other external lines having outgoing 4-momenta p_3^μ and p_4^μ we define as usual $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$. (Hint: you may find the discussion in Chapter 10.2 of Peskin and Schroeder useful, though they use dimensional regularization.)