Homework Set No. 3, Physics 880.08

Deadline – Monday, November 22, 2010

1. In class we quantized free real scalar field theory with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2}\varphi^2.$$

The field operator was shown to be

$$\varphi(x) \; = \; \int \frac{d^3 k}{(2\,\pi)^3\, 2\, E_k} \left[\hat{a}_{\vec{k}}\, e^{-i\,k\cdot x} + \hat{a}^\dagger_{\vec{k}}\, e^{i\,k\cdot x} \right], \tag{1}$$

and the canonical momentum operator was given by $\pi = \dot{\varphi}$. Above $k \cdot x = E_k\, t - \vec{k} \cdot \vec{x}$.

(a) (10 pts) Show that canonical commutation relations

$$\begin{aligned} &[\varphi(\vec{x}, t), \pi(\vec{x}', t)] = i\delta(\vec{x} - \vec{x}') \\ &[\varphi(\vec{x}, t), \varphi(\vec{x}', t)] = 0 \\ &[\pi(\vec{x}, t), \pi(\vec{x}', t)] = 0 \end{aligned} \tag{2}$$

require that the particle creation and annihilation operators obey the following commutation relations

$$\left[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}\right] \; = \; (2\,\pi)^3\, 2\, E_k\, \delta^3(\vec{k} - \vec{k}'), \quad \left[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}\right] \; = \; \left[\hat{a}^\dagger_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}\right] \; = \; 0. \tag{3}$$

(Note that simply showing that Eq. (3) makes the field φ and canonical momentum π operators satisfy canonical commutation relations (2) is not sufficient: you have to show that Eqs. (2) lead to Eqs. (3).)

(b) (5 pts) In class the Hamiltonian for the real scalar field was shown to be

$$\hat{H} \; = \; \int \frac{d^3 k}{(2\,\pi)^3\, 2\, E_k}\, E_k\, \hat{a}^\dagger_{\vec{k}}\, \hat{a}_{\vec{k}}.$$

Find the commutators $[\hat{H}, \hat{a}_{\vec{k}}]$ and $[\hat{H}, \hat{a}^\dagger_{\vec{k}}]$.

(c) (10 pts) Use the results of part **(b)** to write the particle creation and annihilation operators in Heisenberg representation defined by

$$\begin{aligned} \hat{a}^H_{\vec{k}}(t) \; &= \; e^{i\,\hat{H}\,t}\, \hat{a}_{\vec{k}}\, e^{-i\,\hat{H}\,t} \\ \hat{a}^{H\,\dagger}_{\vec{k}}(t) \; &= \; e^{i\,\hat{H}\,t}\, \hat{a}^\dagger_{\vec{k}}\, e^{-i\,\hat{H}\,t} \end{aligned} \tag{4}$$

in terms of E_k, $\hat{a}_{\vec{k}}$ and $\hat{a}^\dagger_{\vec{k}}$ (i.e., eliminate $\hat{H}$ in Eqs. (4) above). Rewrite the field φ from Eq. (1) in terms of the obtained $\hat{a}^H_{\vec{k}}(t)$ and $\hat{a}^{H\,\dagger}_{\vec{k}}(t)$.

2. Consider a free complex scalar theory with the Lagrangian density

$$\mathcal{L} = \partial_\mu \varphi^* \, \partial^\mu \varphi - m^2 \varphi^* \, \varphi. \tag{5}$$

In class we quantized the theory by writing

$$\varphi(x) \; = \; \int \frac{d^3 k}{(2\,\pi)^3 \, 2\, E_k} \, \left[\hat{a}_{\vec{k}} \, e^{-i\, k \cdot x} + \hat{b}^\dagger_{\vec{k}} \, e^{i\, k \cdot x} \right], \tag{6}$$

in terms of two types of operators, $\hat{a}_{\vec{k}}$ and $\hat{b}_{\vec{k}}$.

(a) (8 pts) Express the Hamiltonian for this theory in terms of the operators $\hat{a}_{\vec{k}}$, $\hat{a}^\dagger_{\vec{k}}$, and $\hat{b}_{\vec{k}}$, $\hat{b}^\dagger_{\vec{k}}$, similar to how it was done in class for the real scalar field. You should also drop an infinity, which does not affect any physics in the field theory.

(b) (7 pts) The Lagrangian (5) has a $U(1)$ symmetry with the conserved charge

$$Q \; = \; \int d^3 x \, i \, \left[\varphi^\dagger \, \dot{\varphi} - \dot{\varphi}^\dagger \, \varphi \right] \tag{7}$$

as we showed in class. Using the decomposition from Eq. (6) rewrite the charge Q in terms of $\hat{a}_{\vec{k}}$, $\hat{a}^\dagger_{\vec{k}}$, and $\hat{b}_{\vec{k}}$, $\hat{b}^\dagger_{\vec{k}}$.

3. Consider a Dirac spinor field with the Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi.$$

a. (3 pts) Construct a Hamiltonian $H(t)$ and show that for classical field configurations

$$\frac{d}{dt} H(t) = 0.$$

b. (2 pts) Write $H(t)$ in terms of π and ψ.

c. (5 pts) For quantized field ψ use the anti-commutation relations

$$\left\{ \psi_\alpha(\vec{x}, t), \psi^\dagger_\beta(\vec{x}', t) \right\} = \delta_{\alpha\,\beta} \, \delta(\vec{x} - \vec{x}')$$

along with Dirac equation to show that

$$\begin{aligned} -i\,\partial_0 \psi_\alpha &= [H(t), \psi_\alpha] \\ -i\,\partial_0 \bar{\psi}_\alpha &= \left[H(t), \bar{\psi}_\alpha \right]. \end{aligned}$$