Homework Set No. 3, Physics 880.08

Deadline – Wednesday, May 18, 2011

1. a. (5 pts) By explicitly expanding the exponentials on the left-hand-side and carrying out the Grassmann integrals show that the following relation holds

$$\int d\bar{\chi}_1\, d\chi_1\, d\bar{\chi}_2\, d\chi_2\, \exp\left[-\sum_{i,j} a_{ij}\,\bar{\chi}_i\,\chi_j\right]\, \exp\left[\sum_k (\bar{\chi}_k\,\xi_k + \bar{\xi}_k\,\chi_k)\right] = (\det A)\, \exp\left[\sum_{i,j} \bar{\xi}_i A^{-1}_{ij}\,\xi_j\right]$$

where χ_i and ξ_j are Grassmann variables, and A is a 2×2 Hermitean matrix with elements a_{ij}.

b (5 pts) Show that

$$-i\frac{\partial}{\partial\bar{\xi}}F = \chi\, F = F\,\chi$$
$$i\frac{\partial}{\partial\xi}F = \bar{\chi}\, F = F\,\bar{\chi}$$

for the function

$$F = \exp\left[i\,(\bar{\xi}\,\chi + \bar{\chi}\,\xi)\right].$$

Here χ and ξ are Grassmann variables.

2. Nature of the perturbation series.

Consider a zero-dimensional "field theory" defined by the "path integral"

$$I(m,\lambda) = \int_{-\infty}^{\infty} dx\, e^{-S[x]} \tag{1}$$

where the (Euclidean) action is

$$S[x] = m\,x^2 + \lambda\,x^4.$$

a. (10 pts) Expand $I(m,\lambda)$ in a perturbation series in the powers of λ. Show that the radius of convergence of the series is zero.

You may need the integral definition of the gamma-function

$$\Gamma(z) = \int_0^{\infty} dt\, t^{z-1}\, e^{-t}$$

along with the following property

$$\Gamma(z+1) \,=\, z\,\Gamma(z).$$

b. (10 pts) Let $I_n(m,\lambda)$ denote the truncated perturbation series from part **a** (partial sum) with the highest power of λ being λ^n. Using your favorite numerical software plot $I_n(m=1,\lambda)$ for $n=0,1,2,3,4,5,\ldots$ as functions of λ in the range $\lambda \in [0,0.1]$ (I got better-looking plots in this range, but you may change it to make a better picture). On the same plot draw the curve corresponding to the exact result

$$I(m,\lambda) \,=\, \sqrt{\frac{m}{4\,\lambda}}\; e^{\frac{m^2}{8\,\lambda}}\, K_{1/4}\left(\frac{m^2}{8\,\lambda}\right),$$

with $K_{1/4}$ the modified Bessel function. Demonstrate the asymptotic nature of the series: as you increase the order n, the quality of the perturbative approximation first increases, but then rapidly starts to decrease.

c. OPTIONAL (5 pts) Quasi-classical approximation: evaluate the integral $I(m,\lambda)$ in Eq. (1) using the steepest descent (aka saddle point) method. Find the “classical solution” ($x_{cl}=0$), expand the power of the exponent to quadratic order in fluctuations ξ (where $x = x_{cl}+\xi$), and integrate over all ξ. How good is the approximation? Note that at small-λ the saddle-point approximation works. (This is usually true for field theories too.)

3. (10 pts) Consider a non-Abelian gauge theory with the gauge field A_μ^a and the Lagrangian

$$\mathcal{L} \,=\, -\frac{1}{4}\, F_{\mu\nu}^a\, F^{a\,\mu\nu}.$$

Here

$$F_{\mu\nu}^a \,=\, \partial_\mu\, A_\nu^a - \partial_\nu\, A_\mu^a + g\, f^{abc}\, A_\mu^b\, A_\nu^c$$

with f^{abc} the structure constants of the gauge group $SU(N)$.

Write the equations of motion for this theory. If we define $J^{a\,\mu}$ by

$$\partial_\nu\, F^{a\,\nu\mu} \,=\, J^{a\,\mu}$$

what is $J^{a\,\mu}$ for the above Lagrangian?

4. (10 pts) When constructing gauge-invariant Lagrangian for the non-Abelian gauge field A_μ^a one may consider another Lorentz-invariant

$$I \,=\, \epsilon^{\mu\nu\rho\sigma}\, F_{\mu\nu}^a\, F_{\rho\sigma}^a.$$

Show that this term can be written as a 4-divergence,

$$I = \partial_\mu K^\mu$$

and find the 4-vector K^μ explicitly in terms of the field A_μ^a. Why can not the invariant I serve as the Lagrangian for the non-Abelian field?

(Hint: you may find the identity

$$f^{abe} f^{cde} = \frac{2}{N} (\delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc}) + d^{ace} d^{bde} - d^{bce} d^{ade}$$

useful. Here the gauge group is $SU(N)$ and d^{abc} is the absolutely symmetric object defined by

$$d^{abc} = 2 \operatorname{Tr} \left(T^a \{T^b, T^c\} \right)$$

with T^a the generators of $SU(N)$ in the fundamental representation.)