Homework Set No. 1, Physics 8808.01

Deadline – Thursday, September 6, 2012

1. Consider a real scalar interacting field theory with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\,\phi^2 - \frac{\lambda}{3!}\,\phi^3$$

where λ is a real number.

(a) (5 pts) Construct the Euler-Lagrange equation for this theory.

(b) (5 pts) Find the energy-momentum tensor $T^{\mu\nu}$ for this theory and show explicitly that it is conserved, $\partial_\mu T^{\mu\nu} = 0$, for the fields satisfying the Euler-Lagrange equation found in part (a).

2. The Lagrangian density for a two-component complex scalar field

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

is given by

$$\mathcal{L} = \partial_\mu \vec{\phi}^{\,\dagger} \cdot \partial^\mu \vec{\phi} - m^2\,\vec{\phi}^\dagger\,\vec{\phi}$$

where Hermitean conjugation is defined by

$$\vec{\phi}^{\,\dagger} = (\phi_1^*, \phi_2^*)\,.$$

(a) (3 pts) Show that the above Lagrangian is invariant under the following global $SU(2)$ symmetry

$$\phi_i \to \phi_i' = \left(\exp\left\{i\,\frac{\vec{\alpha}\cdot\vec{\sigma}}{2}\right\}\right)_{ij}\,\phi_j$$

with $\vec{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$ an arbitrary coordinate-independent vector, $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ the Pauli matrices, and $i, j = 1, 2$. Summation over repeated indices is assumed.

(b) (7 pts) Find the conserved current j_μ^a and charge Q^a corresponding to this symmetry (here $a = 1, 2, 3$).

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3. Consider a real scalar field theory with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{\lambda}{4}\,\left(\phi^2 - v^2\right)^2 \tag{1}$$

where λ and v are some positive real numbers. Imagine that the theory lives in $1+1$ space-time dimensions labeled (t, x).

(a) (3 pts) Construct the Hamiltonian for the theory. Show that, for time-independent fields $\phi(t, x) = \phi(x)$, the energy minima (the vacua) of the Hamiltonian are given by

$$\phi = \pm v.$$

(b) (7 pts) Find the time-independent solution of the equations of motion for the Lagrangian (1) that interpolates between the two vacua. That is find the solution $\phi(x)$ satisfying the following conditions

$$\phi(x = -\infty) = -v$$
$$\phi(x = +\infty) = v.$$

(You may also require that $\phi(x = 0) = 0$ for simplicity.) Such solution is known as the 'kink' solution and is the simplest example of a soliton.