Homework Set No. 5, Physics 8808.02

Deadline – Tuesday, March 26, 2013

1. (15 pts) Imagine that the full non-perturbative beta-function of QED were

$$\beta(\alpha) = \frac{\alpha^2}{3\pi}\left[1 - e^{1-\frac{1}{\alpha}}\right].$$

Find the running QED coupling constant $\alpha(Q^2)$ for such beta-function. Sketch $\alpha(Q^2)$ as a function of Q^2. Find the UV fixed point and determine the large-Q^2 asymptotics of $\alpha(Q^2)$, i.e., find how it approaches the fixed point.

2. a. (15 pts) Consider a harmonic oscillator in a background of a time-dependent external force (source) $j(t)$. The Lagrangian is

$$L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2 + q\, j(t).$$

Using quasi-classical method for evaluation of path integrals find the time-evolution (Feynman) kernel

$$\begin{aligned} U(q_f, t_f; q_i, t_i) &= {}_S\langle q_f(t_f)|\, e^{-\frac{i}{\hbar}\hat{H}\,(t_f - t_i)}\, |q_i(t_i)\rangle_S = {}_H\langle q_f, t_f | q_i, t_i\rangle_H \\ &= \int [\mathcal{D}q]\, \exp\left\{ \frac{i}{\hbar} \int\limits_{t_i}^{t_f} dt\, L(t) \right\}. \end{aligned}$$

You may use the result derived in class for the harmonic oscillator without the external force, though this time you also need to find the classical action in terms of $j(t)$. In evaluating the classical action assume that $q_i = q_f = 0$, or, more specifically, require that $q_{cl}(t) = 0$ when $j(t) = 0$. When solving classical EOM you may find Fourier-integral decomposition

$$q(t) = \int\limits_{-\infty}^{\infty} \frac{dE}{2\pi}\, e^{-\frac{i}{\hbar} E\, t}\, q_E$$

useful.

b. (10 pts) Use the result of part **a** to show that the two-point function for the harmonic oscillator without the external force is given by

$$\langle 0 |\, \mathrm{T}\, \hat{q}(t_1)\, \hat{q}(t_2)\, |0\rangle = i\, \frac{\hbar^2}{m} \int\limits_{-\infty}^{\infty} \frac{dE}{2\pi}\, \frac{e^{-\frac{i}{\hbar} E\,(t_1 - t_2)}}{E^2 - \hbar^2\omega^2 + i\,\epsilon}. \qquad (1)$$

c. (10 pts) Re-derive the two-point function in Eq. (1) by using creation and annihilation operators. For the simple harmonic oscillator (without the external force) write

$$\hat{q}(t) = \sqrt{\frac{\hbar}{2\,m\,\omega}}\,\left[\hat{a}\,e^{-i\,\omega\,t} + \hat{a}^\dagger\,e^{i\,\omega\,t}\right]$$

and use commutation relations for $\hat{a}$ and $\hat{a}^\dagger$ ($[\hat{a}, \hat{a}^\dagger] = 1$, all other commutators are zero) to obtain Eq. (1).