Homework Set No. 3, Physics 8808.1

Deadline – Thursday, October 4, 2012

1. (10 pts) In class we showed that Dirac spinors transform as

$$\psi_D(x) \;\to\; \psi'_D(x') \;=\; \begin{pmatrix} e^{-\frac{i}{2}\,\vec{\sigma}\cdot(\vec{\theta}+i\vec{\xi})} & 0 \\ 0 & e^{-\frac{i}{2}\,\vec{\sigma}\cdot(\vec{\theta}-i\vec{\xi})} \end{pmatrix} \psi_D(x) \qquad (1)$$

under Lorentz transformations. Show that this transformation rule is equivalent to

$$\psi_D(x) \;\to\; \psi'_D(x') \;=\; e^{-\frac{i}{4}\,\omega^{\mu\,\nu}\,\sigma_{\mu\,\nu}}\,\psi_D(x)$$

where

$$\sigma_{\mu\,\nu} \;=\; \frac{i}{2}\,[\gamma_\mu, \gamma_\nu].$$

As usual $\xi^i = \omega^{0i}$ and $\theta_i = \frac{1}{2}\,\epsilon_{ijk}\,\omega_{jk}$. You may consider boosts and rotations separately for the full credit.

2. (a) (5 pts) Complete the proof started in class that

$$\bar{\psi}\,\gamma^\mu\,\psi$$

is a 4-vector by showing that it transforms like one under infinitesimal boosts. ψ is the Dirac spinor which transforms according to Eq. (1).

(b) (5 pts) Prove that

$$\bar{\psi}\,\gamma^\mu\,\gamma^5\,\psi$$

is a 4-vector under both boosts and rotations. What happens to it under parity?

3. Consider a massive Dirac field ψ with mass m.

a. (2 pts) Starting with Dirac equation

$$[i\gamma^\mu \partial_\mu - m]\psi = 0$$

derive an equation for $\bar{\psi}$.

b. (3 pts) Using the result of part **a** show that the electromagnetic current

$$j_\mu = \bar{\psi}\gamma_\mu\psi$$

is conserved at the classical level, i.e., show that $\partial_\mu j^\mu = 0$.

c. (5 pts) Defining $\gamma^5 = i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3$ use the anti-commutation relations for γ-matrices $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ to show that

$$\{\gamma^\mu, \gamma^5\} = 0.$$

Use this result to show that the divergence of the axial vector current

$$j^{5\,\mu} = \bar{\psi}\gamma^\mu\gamma^5\psi$$

is

$$\partial_\mu\, j^{5\,\mu} \;=\; 2\,i\,m\,\bar{\psi}\,\gamma^5\,\psi.$$

That is the axial current is conserved for massless particles (in this classical theory).

4. Problem **3.4** (a, b, c) from Peskin and Schroeder. Each part is worth 5 points.

You may assume that it is known that $\chi_L^\dagger \bar{\sigma}^\mu \chi_L$ transforms as a 4-vector, since it was proved in class and in problem 2 above. You may also find the following relation useful (Eq. (3.38) in Peskin and Schroeder):

$$\sigma^2\,(\sigma^i)^* = -\sigma^i\,\sigma^2$$

along with a compact notation for the Dirac γ's

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}.$$