Homework Set No. 5, Physics 8808.01

Deadline – Thursday, November 1, 2012

1. Time-ordered product of real scalar fields is defined by

$$\mathrm{T}\phi(x)\,\phi(y) \;=\; \theta(x^0 - y^0)\,\phi(x)\,\phi(y) + \theta(y^0 - x^0)\,\phi(y)\,\phi(x),$$

where ϕ's are operators in Heisenberg picture.

a. (5 pts) In a free scalar field theory with mass m use Klein-Gordon equation along with the canonical commutation relations to show that

$$[\partial^2 + m^2]\,\mathrm{T}\phi(x)\,\phi(y) \;=\; -i\,\delta^4(x-y)$$

where the derivative squared (the D'Alembertian) is taken with respect to 4-coordinates x.

b. (10 pts) Similar to what we did in class for retarded Green function, find an explicit expression for the Feynman propagator in coordinate space in a massless ($m = 0$) theory by performing the following integral

$$D_F(x-y) \;=\; \int \frac{d^4k}{(2\,\pi)^4}\, e^{-i\,k\cdot(x-y)}\, \frac{i}{k^2 + i\,\epsilon}.$$

Is Feynman propagator causal?

2. Time-ordered product of Dirac spinors is defined by

$$\mathrm{T}\,\psi_\alpha(x)\,\bar{\psi}_\beta(y) \;=\; \theta(x^0 - y^0)\,\psi_\alpha(x)\,\bar{\psi}_\beta(y) - \theta(y^0 - x^0)\,\bar{\psi}_\beta(y)\,\psi_\alpha(x)$$

where $\alpha, \beta = 1, 2, 3, 4$ are Dirac indices. ψ and $\bar{\psi}$ are operators in Heisenberg picture.

a. (3 pts) In a free Dirac field theory with mass m use Dirac equation and canonical anti-commutation relations to show that (α' is summed over)

$$[i\,\gamma^\mu\,\partial_\mu - m]_{\alpha\,\alpha'}\,\mathrm{T}\,\psi_{\alpha'}(x)\,\bar{\psi}_\beta(y) \;=\; i\,\mathbf{1}_{\alpha\beta}\,\delta^4(x-y).$$

b. (10 pts) Perform explicit calculation to show that the Feynman propagator for fermions is

$$S_F(x-y) \;\equiv\; \langle 0|\mathrm{T}\,\psi(x)\,\bar{\psi}(y)|0\rangle \;=\; \int \frac{d^4k}{(2\,\pi)^4}\, e^{-i\,k\cdot(x-y)}\, \frac{i\,(\gamma\cdot k + m)}{k^2 - m^2 + i\,\epsilon}.$$

(We did much of this in class, but it is still useful to do by yourself.)

3. a. (3 pts) For a free real scalar field theory with mass m show that

$$[\phi(x), \phi(y)] = \int \frac{d^4k}{(2\pi)^4} e^{-i\,k\cdot(x-y)}\,\text{Sign}(k^0)\,(2\pi)\,\delta(k^2-m^2) \tag{1}$$

where

$$\text{Sign}(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0. \end{cases}$$

Again ϕ's are operators in Heisenberg picture, x and y are some coordinate space 4-vectors.

b. (2 pts) Verify that Eq. (1) gives correct equal-time commutation relations for the fields

$$[\phi(\vec{x}, t), \phi(\vec{y}, t)] = 0.$$

c. (5 pts) Using Eq. (1) show that

$$\left(\frac{\partial}{\partial x^0} [\phi(x), \phi(y)] \right)_{y^0=x^0} = -i\,\delta^3(\vec{x}-\vec{y}),$$

i.e., reproduce another one of the equal-time canonical commutation relations. Also show that

$$[\dot{\phi}(\vec{x}, t), \dot{\phi}(\vec{y}, t)] = 0$$

reproducing the remaining canonical commutation relation.

4. (7 pts) Construct scalar electrodynamics following steps similar to what we did in class in deriving QED. First consider a complex scalar field theory with the Lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \,\partial^\mu \phi - m^2\,\phi^*\,\phi.$$

It is invariant under a global $U(1)$ symmetry: $\phi(x) \to e^{i\,\alpha}\,\phi(x)$ with α a real number. Gauge this Lagrangian by modifying it to have a *local* $U(1)$ symmetry

$$\phi(x) \to e^{i\,\alpha(x)}\,\phi(x) \tag{2}$$

with the help of a new vector field A_μ which transforms as

$$A_\mu \to A_\mu - \frac{1}{e}\,\partial_\mu\,\alpha(x). \tag{3}$$

Finally add the Lagrangian density for the free vector field. What is the resulting Lagrangian? Show explicitly that it is invariant under the local $U(1)$ transformations given by Eqs. (2) and (3).