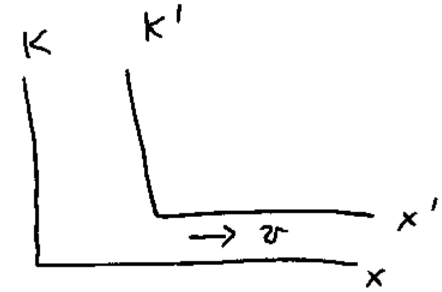
Classical Field Theory

4-vectors, notations



defining $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$
write Lorentz transformation

as
$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \quad \beta = v/c, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

Definition: a 4-vector $A^\mu = \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$ is an object

which under Lorentz transformation transforms

as
$$\begin{pmatrix} A'^0 \\ A'^1 \\ A'^2 \\ A'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}.$$

(example: x^μ is a contravariant 4-vector)

$\Rightarrow$ A^μ is a contravariant vector: $A'^\mu = \dfrac{\partial x'^\mu}{\partial x^\nu} A^\nu$

B_μ is a covariant vector: $B'_\mu = \dfrac{\partial x^\nu}{\partial x'^\mu} B_\nu$

$\Rightarrow$ $\dfrac{\partial \varphi}{\partial x^\mu} \equiv \partial_\mu \varphi$ with φ scalar field is a covariant vector as $\dfrac{\partial \varphi}{\partial x'^\mu} = \dfrac{\partial x^\nu}{\partial x'^\mu} \dfrac{\partial \varphi}{\partial x^\nu}$.

Tensors: $A^\mu B^\mu$ ~ contravariant, $A_\mu B_\nu$ ~ covariant (rank 2), can have higher ranks.

Def. Scalar (inner) product of 2 vectors is $A_\mu B^\mu$. (assume summation).

It is Lorentz-invariant: $A'_\mu B'^\mu = \dfrac{\partial x^\alpha}{\partial x'^\mu} A_\alpha \dfrac{\partial x'^\mu}{\partial x^\beta} B^\beta$

$= \dfrac{\partial x^\alpha}{\partial x^\beta} A_\alpha B^\beta = \delta^\alpha_\beta A_\alpha B^\beta = A_\alpha B^\alpha$.

Def. The interval $ds^2 \left(= dx_\mu dx^\mu\right) = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$.

It's a Lorentz-invariant too.

Def. The metric tensor $g_{\mu\nu}$ is defined by

$$\boxed{ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu}$$

In our Minkowski space (throughout the course we'll use this notation) $g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \eta_{\mu\nu}$

$dx_\mu dx^\mu$ ~ also a Lorentz-scalar $\Rightarrow dx_\mu = g_{\mu\nu} dx^\nu$

$\Rightarrow g_{\mu\nu}$ lowers & raises indeces!

Example: $x^\mu = (ct, \vec{x}) \Rightarrow x_\mu = g_{\mu\nu} x^\nu = (ct, -\vec{x})$
contravariant covariant

In general $A_\mu = g_{\mu\nu} A^\nu$, $A^\mu = g^{\mu\nu} A_\nu$

where $g^{\mu\nu}$ is defined by requiring that

$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu{}_\nu$: if that is true $\Rightarrow$ start

with $A_\mu = g_{\mu\nu} A^\nu \Rightarrow g^{\alpha\mu} A_\mu = g^{\alpha\mu} g_{\mu\nu} A^\nu =$

$= \delta^\alpha{}_\nu A^\nu = A^\alpha \Rightarrow A^\alpha = g^{\alpha\mu} A_\mu$.

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \eta^{\mu\nu} \text{ too. } \quad (= g_{\mu\nu})$$

Def. $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}$, $\partial^\mu \equiv \dfrac{\partial}{\partial x_\mu} \Rightarrow \partial_\mu \varphi$ is a covariant vector,

$\partial^\mu \varphi$ is a contravariant vector. (check!)

$\partial_\mu A^\mu$ is a Lorentz-invariant.

$\partial_\mu \partial^\mu = \dfrac{1}{c^2} \dfrac{\partial^2}{\partial t^2} - \vec{\nabla}^2$ is also Lorentz-invariant.

Examples: other important 4-vectors are

$$p^\mu = \begin{pmatrix} \mathcal{E}/c \\ \vec{p} \end{pmatrix}, \quad p_\mu = \begin{pmatrix} \mathcal{E}/c \\ -\vec{p} \end{pmatrix} \Rightarrow p_\mu p^\mu = \left(\frac{\mathcal{E}}{c}\right)^2 - \vec{p}^{\,2} = m^2 c^2.$$

$A^\mu = (\Phi, \vec{A})$ in E&M, Φ ~ electric potential,

$\vec{A}$ - vector potential.

$J^\mu = (c\rho, \vec{J})$ with ρ the charge density, $\vec{J}$ the current density.

Notations from now on $\boxed{c = 1}$ and $\boxed{\hbar = 1}$

"natural units".

$\Rightarrow$ mass, momentum, energy are measured in

the same units (eV, keV, MeV, GeV, ...)

$$1\,eV = 1.6 \times 10^{-19}\,J$$

distances, time are measured in femto-

-meters aka fermis (fm):

$$1\,fm \simeq 5\,GeV^{-1}$$

$1\,GeV = 10^9\,eV$, 1 femto-meter $= 10^{-15}\,m. = 1\,fm.$

proton's mass $m_p = 0.938\,GeV \simeq 1\,GeV$

electron's mass $m_e = 0.511\,MeV \simeq 0.5 \times 10^{-3}\,GeV$

Classical Scalar Field Theory (real field)

$\varphi(x^\mu) = \varphi(x^0, \vec{x})$ ~ a function of space-time points x^μ

(example ~ temperature field $T(t, \vec{x})$)

in Classical Mechanics one has point particles $i = 1, \dots, N$ with the Lagrangian $L(q_i, \dot{q}_i, t)$

and the action $S = \int dt \, L(q_i, \dot{q}_i, t)$

q_i ~ degrees of freedom (e.g. particle coordinates)

$\dot{q}_i = \frac{dq_i}{dt}$ ~ generalized velocities

Now, instead of discrete point particles we have a field $\varphi(\vec{x}, t)$ $\Rightarrow$

Classical Mechanics		Classical Field Theory
q_i	$\longrightarrow$	$\varphi(x^0, \vec{x})$
i	$\longrightarrow$	$\vec{x}, t$
$\dot{q}_i$	$\longrightarrow$	$\partial_\mu \varphi$, $\mu = 0, 1, 2, 3$
$L(q_i, \dot{q}_i, t)$	$\longrightarrow$	$\int d^3x \, \mathcal{L}(\varphi, \partial_\mu \varphi)$

$\mathcal{L}$ is <u>Lagrangian density</u>. (usually called the lagrangian)

The action is $S = \int dt\, L = \underbrace{\int dt\, d^3x}_{d^4x}\, \mathcal{L}(\varphi, \partial_\mu \varphi)$ (remember $c=1$)

S is a Lorentz-scalar (better be, physics is Lorentz-invariant)

What about $d^4x = dx^0 dx^1 dx^2 dx^3$? Remember that

$x'^\mu = \Lambda^\mu{}_\nu x^\nu$ with $\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$ a matrix of L. tr.

$\Rightarrow d^4x' = \underbrace{\det \Lambda}_{\text{Jacobian}} \cdot d^4x$

Now, $\det \Lambda = \det \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \gamma^2(1-\beta^2) = 1$ (true in general)

$\Rightarrow d^4x' = d^4x \Rightarrow d^4x$ is a Lorentz-scalar

$\Rightarrow$ $\mathcal{L}$ is <u>a Lorentz-scalar</u>!

Just like in classical mechanics, in classical field theory dynamics is given by the <u>least action principle</u>: field φ is determined by requiring that S is stationary with respect to small perturbation around φ : $S[\varphi + \delta\varphi] = S[\varphi] + o(\delta\varphi^2)$.

$$0 = \delta S = \int d^4x \left[\frac{\delta \mathcal{L}}{\delta \varphi} \delta\varphi + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} \delta(\partial_\mu \varphi) \right] =$$

$$= (\text{as } \delta \partial_\mu \varphi = \partial_\mu \delta\varphi \Rightarrow \text{parts}) = \int d^4x \left[\frac{\delta \mathcal{L}}{\delta\varphi} \delta\varphi - \right.$$

$$\left. - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} \right) \delta\varphi \right] + \underbrace{\text{surface term}}_{0}$$

$$\Rightarrow 0 = \int d^4x \, \delta\varphi \left[\frac{\delta\mathcal{L}}{\delta\varphi} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} \right) \right] \quad \text{for any } \delta\varphi$$

$$\Rightarrow \boxed{\frac{\delta \mathcal{L}}{\delta\varphi} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} \right) = 0}$$

Euler - Lagrange equations (aka equations of motion) for field φ. (EOM)

Now, $\varphi(x)$ is a scalar field $\Rightarrow$ it is Lorentz-inv., which means that : $\varphi(x) \to \varphi'(x') = \varphi(x)$

$\Rightarrow$ as $x'^\mu = \Lambda^\mu{}_\nu x^\nu \Rightarrow x' = \Lambda \cdot x \Rightarrow \varphi'(x) = \varphi(\Lambda^{-1} x)$.

Lagrangian density for massive scalar field:

$$\boxed{\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \, \partial^\mu \varphi - \frac{m^2}{2} \varphi^2}$$

EOM: $\dfrac{\delta \mathcal{L}}{\delta\varphi} = -m^2 \varphi$; $\dfrac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} = \partial^\mu \varphi \Rightarrow$

$$\Rightarrow \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} \right) = \partial_\mu \partial^\mu \varphi \Rightarrow -m^2\varphi - \partial_\mu \partial^\mu \varphi = 0$$

$$\Rightarrow \quad \boxed{\left[\partial_\mu \partial^\mu + m^2\right] \varphi = 0} \qquad \text{Klein-Gordon equation}$$

or $\boxed{\left[\Box + m^2\right] \varphi = 0}$

To solve K-G equation write $\varphi(x) = \int d^4k \, e^{-ik\cdot x} \tilde{\varphi}(k)$

with $k \cdot x = k_\mu x^\mu = k^0 x^0 - \vec{k}\cdot\vec{x}$.

$$\left[\Box + m^2\right] \varphi = \int d^4k \, \tilde{\varphi}(k) \left(\Box + m^2\right) e^{-ik\cdot x} = \int d^4k \, \tilde{\varphi}(k) \cdot$$

$$\cdot \left[-k^2 + m^2\right] = 0 \qquad \text{with } k^2 = k_\mu k^\mu = (k^0)^2 - (\vec{k})^2.$$

$\Rightarrow \left[k^2 - m^2\right] \tilde{\varphi} = 0 \Rightarrow$ as $\tilde{\varphi} \neq 0 \Rightarrow k^2 = m^2$ or

$E_k^2 - \vec{k}^2 = m^2 \Rightarrow \mathcal{E}_k = \pm\sqrt{\vec{k}^2 + m^2} \Rightarrow$ define $\boxed{\mathcal{E}_k = \sqrt{\vec{k}^2 + m^2}}$

$$\Rightarrow \boxed{\varphi(x) = \int \frac{d^3k}{(2\pi)^3 \, 2\mathcal{E}_k} \left[a_{\vec{k}} \, e^{-i\mathcal{E}_k t + i\vec{k}\cdot\vec{x}} + a^*_{\vec{k}} \, e^{i\mathcal{E}_k t - i\vec{k}\cdot\vec{x}} \right]}$$

most general solution