Last time: Classical Field Theory (cont'd)

reviewed 4-vectors: $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$ Lorentz transform

contravariant:

$$A'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} A^{\nu}$$

covariant:

$$B'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} B_{\nu}$$

Defined: scalar product $A_{\mu} B^{\mu} = A_0 B^0 + A_1 B^1 + A_2 B^2 + A_3 B^3$.
(a Lorentz invariant)

metric tensor $ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \equiv g_{\mu\nu} dx^{\mu} dx^{\nu}$

$\Rightarrow$ $g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$; $g^{\mu\alpha} g_{\alpha\nu} = \delta^{\mu}{}_{\nu}$ ~ definition of $g^{\mu\alpha}$.

Showed that: $A_{\mu} = g_{\mu\nu} A^{\nu}$, $A^{\mu} = g^{\mu\nu} A_{\nu}$

Defined: derivatives $\partial_{\mu} \varphi = \frac{\partial}{\partial x^{\mu}} \varphi$; $\partial^{\mu} \varphi = \frac{\partial}{\partial x_{\mu}} \varphi$.

Examples: $\partial_{\mu} A^{\mu}$ ~ L. inv, $\partial_{\mu} \partial^{\mu}$ ~ same $\Box = \partial_{\mu} \partial^{\mu}$

$p^{\mu} = \left(\frac{E}{c}, \vec{p} \right)$, $p_{\mu} = \left(\frac{E}{c}, -\vec{p} \right)$ ~ 4-momentum

put: $c = 1$ $\hbar = 1$

Classical Scalar Field Theory (cont'd)

$\varphi(x)$ ~ scalar field: $\varphi(x) \xrightarrow{\mathcal{L}} \varphi'(x') = \varphi(x)$ where $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$

The action: $S = \int d^4x \, \mathcal{L}(\varphi, \partial_{\mu} \varphi)$

Lagrangian density

Least action principle: $\boxed{\delta S = 0}$

leads to $\boxed{\frac{\delta \mathcal{L}}{\delta \varphi} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} \right) = 0}$ Euler-Lagrange eqns

EOM = Equations of motion

We wrote the Lagrangian for free real massive scalar field:

(1) $\mathcal{L}$ ~ Lorentz invariant

(2) no high-order derivatives

(3) free field = linear EOM $\Rightarrow$ no powers higher than φ^2 $\Rightarrow \mathcal{L} = A(\varphi)\, \partial_\mu \varphi \partial^\mu \varphi + B(\varphi)$

$= \frac{1}{2} \partial_\mu \varphi' \partial^\mu \varphi' - U(\varphi)$ if $\sqrt{A}\, d\varphi = \frac{1}{\sqrt{2}} d\varphi'$

$$\Rightarrow \quad \mathcal{L} = \frac{1}{2} \partial_\mu \varphi \, \partial^\mu \varphi - \frac{m^2}{2} \varphi^2$$

Build EOM: $\frac{\delta \mathcal{L}}{\delta \varphi} = -\frac{m^2}{2} \underbrace{\frac{\delta \varphi^2}{\delta \varphi}}_{= 2\varphi} = -m^2 \varphi$

$$\frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} = \frac{\delta \left(\frac{1}{2} \partial_\alpha \varphi \, \partial^\alpha \varphi \right)}{\delta (\partial_\mu \varphi)} = \frac{\delta \left(\frac{1}{2} g^{\alpha\beta} (\partial_\alpha \varphi)(\partial_\beta \varphi) \right)}{\delta (\partial_\mu \varphi)} =$$

$$= \left| \text{ as } \frac{\delta(\partial_\alpha \varphi)}{\delta(\partial_\mu \varphi)} = \delta^\mu_\alpha \right. = \frac{1}{2} g^{\alpha\beta} \delta^\mu_\alpha \partial_\beta \varphi + \frac{1}{2} g^{\alpha\beta} (\partial_\alpha \varphi) \delta^\mu_\beta$$

$$= \frac{1}{2} g^{\mu\beta} (\partial_\beta \varphi) + \frac{1}{2} g^{\alpha\mu} (\partial_\alpha \varphi) = \frac{1}{2} \partial^\mu \varphi + \frac{1}{2} \partial^\mu \varphi = \partial^\mu \varphi.$$

$\Rightarrow$ EOM is $-m^2 \varphi - \partial_\mu \partial^\mu \varphi = 0 \Rightarrow \boxed{(\partial_\mu \partial^\mu \varphi + m^2) \varphi = 0}$

or $[\Box + m^2] \varphi = 0$ Klein-Gordon equation

$$\Rightarrow \boxed{\left[\partial_\mu \partial^\mu + m^2\right] \varphi = 0}$$ Klein-Gordon equation

or $$\boxed{\left[\Box + m^2\right] \varphi = 0}$$

To solve K-G equation write $\varphi(x) = \int d^4k \, e^{-ik\cdot x} \tilde{\varphi}(k)$

with $k \cdot x = k_\mu x^\mu = k^0 x^0 - \vec{k}\cdot\vec{x}$.

$$\left[\Box + m^2\right] \varphi = \int d^4k \, \tilde{\varphi}(k) \left(\Box + m^2\right) e^{-ik\cdot x} = \int d^4k \, \tilde{\varphi}(k) \cdot$$

$$\cdot \left[-k^2 + m^2\right] = 0 \quad \text{with } k^2 = k_\mu k^\mu = (k^0)^2 - (\vec{k})^2.$$

$\Rightarrow \left[k^2 - m^2\right] \tilde{\varphi} = 0 \Rightarrow$ as $\tilde{\varphi} \neq 0 \Rightarrow k^2 = m^2$ or

$E_k^2 - \vec{k}^2 = m^2 \Rightarrow \mathcal{E}_k = \pm\sqrt{\vec{k}^2 + m^2} \Rightarrow$ define $\boxed{\mathcal{E}_k = \sqrt{\vec{k}^2 + m^2}}$

$$\Rightarrow \boxed{\varphi(x) = \int \frac{d^3k}{(2\pi)^3\, 2\mathcal{E}_k} \left[a_{\vec{k}}\, e^{-i\mathcal{E}_k t + i\vec{k}\cdot\vec{x}} + a^*_{\vec{k}}\, e^{i\mathcal{E}_k t - i\vec{k}\cdot\vec{x}} \right]}$$

most general solution.

One in general would write

$$\tilde{\varphi}(k) \propto (2\pi)\, \delta(k^2 - m^2)\, \theta(k^0)$$

pick physical (positive energy) solution

$$\Rightarrow \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \tilde{\varphi}(k) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} (2\pi)\, \delta(k^2 - m^2)\, \theta(k^0).$$

$$\cdot\, a_{\vec{k}} = \int \frac{d^3k}{(2\pi)^3} dk^0\, e^{-ik^0 t + i\vec{k}\cdot\vec{x}}\, \delta((k^0)^2 - \vec{k}^2 - m^2).$$

coefficient

$$\cdot\, \theta(k^0)\, a_{\vec{k}} = \int \frac{d^3k}{(2\pi)^3} e^{-i\mathcal{E}_k t + i\vec{k}\cdot\vec{x}} \frac{1}{2\mathcal{E}_k} a_{\vec{k}} =$$

$$= \int \underbrace{\frac{d^3k}{2\mathcal{E}_k (2\pi)^3}}_{\text{boost invariant phase space measure}} e^{-i\mathcal{E}_k t + i\vec{k}\cdot\vec{x}}\, a_{\vec{k}}$$

$\Rightarrow$ to get full real φ add c.c. $\Rightarrow$

$$\Rightarrow \boxed{\varphi = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \left[a_{\vec{k}}\, e^{-i\mathcal{E}_k t + i\vec{k}\cdot\vec{x}} + a^*_{\vec{k}}\, e^{i\mathcal{E}_k t - i\vec{k}\cdot\vec{x}} \right]}.$$

<u>problem:</u> have negative energy modes, having $-\mathcal{E}_k$.
If φ is a wave function $\sim$ have negative energy states $\sim$ bad.

Conservation Laws & Noether's Theorem.

Noether's theorem | Every symmetry (of S) gives a conservation law.

If $S \to S' = S$ when $\phi \to \phi'$, $x^\mu \to x'^\mu$ $\Rightarrow$ there exists one or more conserved quantities.

Example 1 | Consider complex (!) scalar field $\phi(x)$ with $\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi^* - m^2 |\phi|^2$.

It is invariant under $\phi \to e^{i\alpha}\phi$, $\phi^* \to e^{-i\alpha}\phi^*$ with α a real constant. ($U(1)$ symmetry group)

$$0 \overset{\mathcal{L}\text{ is inv.}}{=} \delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta(\partial_\mu\phi) + \frac{\delta\mathcal{L}}{\delta\phi^*}\delta\phi^*$$

$$+ \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^*)}\delta(\partial_\mu\phi^*) = \underbrace{\left[\frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right)\right]}_{=0}\delta\phi +$$

$$+ \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi\right) + \underbrace{\left[\frac{\delta\mathcal{L}}{\delta\phi^*} - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^*)}\right)\right]}_{=0}\delta\phi^* +$$

$$+ \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^*)}\delta\phi^*\right)$$

$[\dots] = 0$ by Euler-Lagrange equations

$$\Rightarrow \quad 0 = \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta\phi + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^*)} \delta\phi^* \right]$$

Infinitesimal transform: $\phi \to e^{i\delta\alpha}\phi \simeq (1 + i\delta\alpha)\phi$

$$\Rightarrow \quad \delta\phi = i\,\delta\alpha\,\phi \quad ; \quad \delta\phi^* = -i\,\delta\alpha\,\phi^*$$

$$\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} = \partial^\mu \phi^* \quad ; \quad \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^*)} = \partial^\mu \phi$$

$$\Rightarrow \quad 0 = i\,\delta\alpha\;\partial_\mu \underbrace{\left[\phi\,\partial^\mu \phi^* - \phi^*\,\partial^\mu \phi \right]}_{\text{call this } j^\mu}$$

$\Rightarrow$ $\boxed{j^\mu = i\left[\phi\,\partial^\mu \phi^* - \phi^*\,\partial^\mu \phi\right]}$ is a <u>conserved current</u>

as $\partial_\mu j^\mu = 0$.

In general if $S \to S' = S$ under $\phi \to \phi'$ $\Rightarrow$ as S is inv.

$$\Rightarrow \quad \mathcal{L} \to \mathcal{L}' = \mathcal{L} + \delta\alpha \cdot \underbrace{\partial_\mu J^\mu}$$

4-divergence ~ surface term in S.

$\Rightarrow$ straightforward to find J^μ

(if $\mathcal{L} = \mathcal{L}'$ $\Rightarrow$ J^μ is conserved)