Last time: Free scalar field theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2}\varphi^2$$

EOM: $[\partial_\mu \partial^\mu + m^2]\varphi = 0$ Klein-Gordon eqn

Found the solution of KG eqn:

$$\varphi = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k}\left[a_{\vec{k}}\, e^{-i\mathcal{E}_k t + i\vec{k}\cdot\vec{x}} + a^*_{\vec{k}}\, e^{i\mathcal{E}_k t - i\vec{k}\cdot\vec{x}}\right]$$

where $\sqrt{\vec{k}^2 + m^2} = \mathcal{E}_K$, $a_{\vec{k}}$ ~ an arbitrary function

problem: states with $-\mathcal{E}_K < 0$ energy!

Conservation Laws & Noether's Theorem (cont'd)

Noether's Th'm: if $S \to S' = S$ under $\phi \to \phi'$, $x^\mu \to x'^\mu$

$\Rightarrow$ there exists a conservation law

Showed that $\delta\mathcal{L} = \sum_a \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^a)}\delta\phi^a \right]$ under $\varphi \to \varphi + \delta\varphi$

if we have several scalar fields ϕ^a, $a = 1, \ldots, N$.

If $\mathcal{L} \to \mathcal{L}' = \mathcal{L} \Rightarrow \delta\mathcal{L} = 0 \Rightarrow j^\mu \propto \sum_a \frac{\delta\mathcal{L}}{\delta(\partial_\mu \varphi^a)}\delta\varphi^a$

such that $\partial_\mu j^\mu = 0$ ~ conserved current.

For complex scalar field we had:

Example 1

$$\mathcal{L} = \partial_\mu \varphi \, \partial^\mu \varphi^* - m^2 |\varphi|^2$$

$$\Rightarrow j^\mu = i \left[\varphi \, \partial^\mu \varphi^* - \varphi^* \partial^\mu \varphi \right]$$ conserved current

$$\partial_\mu j^\mu = 0.$$

Mentioned conserved charges:

$$Q_{(t)} = \int d^3x \; j^0(\vec{x}, t) \qquad \frac{dQ(t)}{dt} = 0 \quad \text{(proved)}$$

$$\Rightarrow \left\{ Q = \int d^3x \left[\varphi \, \partial^0 \varphi^* - \varphi^* \partial^0 \varphi \right] \right\}$$

for complex scalar field.

Comlex φ: j^μ can be interpreted as EM current

$Q \sim$ EM charge!

If φ is <u>real</u> $\Rightarrow$ $j^\mu = 0$, $Q = 0$ no charge $\Rightarrow$

$\Rightarrow$ need complex (or multi-component) φ to have EM charge.

Example 2 | Imagine a theory with

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi) \qquad (\text{no } x^\mu\text{-dependence in } \mathcal{L})$$

Imagine an infinitesimal space-time shift:

$$x^\mu \to x^\mu - \delta a^\mu = x'^\mu \Rightarrow x^\mu = x'^\mu + \delta a^\mu$$

$$\Rightarrow \phi(x) \xrightarrow{\phi \text{ is inv.}} \phi(x'^\mu + \delta a^\mu) \simeq \phi(x'^\mu) + \delta a^\mu \partial_\mu \phi(x')$$

$$\delta \mathcal{L} = \frac{\delta \mathcal{L}}{\delta \phi}\delta\phi + \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\delta(\partial_\mu \phi)\right) = \underbrace{\left[\frac{\delta \mathcal{L}}{\delta \phi} - \partial_\mu\left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\right)\right]}_{\overset{\shortparallel}{0}\ (\text{EOM})}\delta\phi$$

$$+ \partial_\mu\left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\delta\phi\right) = \partial_\mu\left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\delta\phi\right)$$

$$\Rightarrow \delta \mathcal{L} = \partial_\mu\left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\delta a^\nu \partial_\nu \phi\right).$$

On the other hand $\mathcal{L}$ is scalar $\Rightarrow \mathcal{L} = \mathcal{L}(x) \Rightarrow$

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \delta a^\mu \partial_\mu \mathcal{L} \Rightarrow \delta \mathcal{L} = \delta a^\nu \partial_\mu(\delta^\mu_\nu \mathcal{L})$$

Equating two $\delta\mathcal{L}$'s we get

$$\delta a^\nu \partial_\mu\left[\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\partial_\nu \phi - \delta^\mu_\nu \mathcal{L}\right] = 0$$

Def. Energy-momentum tensor

$$\boxed{T^{\mu}{}_{\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \partial_\nu \phi - \delta^{\mu}{}_{\nu} \mathcal{L}}$$

$\Rightarrow$ $\partial_\mu T^{\mu}{}_{\nu} = 0$ conserved!

For $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi$ get $\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} = \partial^\mu \phi$

$$\Rightarrow T^{\mu}{}_{\nu} = \partial^\mu \phi \, \partial_\nu \phi - \delta^{\mu}{}_{\nu} \frac{1}{2} \partial_\rho \phi \partial^\rho \phi .$$

$$\Rightarrow T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\rho \phi \partial^\rho \phi$$

but: not always symmetric (will see more later)

<u>Conserved charges:</u> for a conserved current j^μ

(such that $\partial_\mu j^\mu = 0$) we have a charge:

$$Q(t) = \int d^3x \; j^0(\vec{x}, t)$$

(e.g. electric charge).

$$\frac{dQ(t)}{dt} = \int d^3x \, \frac{\partial}{\partial t} j^0(\vec{x},t) = \int d^3x \left[\underbrace{\partial_\mu j^\mu}_{=0} - \vec{\nabla} \cdot \vec{j} \, \right] =$$

$$= - \int d^3x \, \vec{\nabla} \cdot \vec{j} \overset{\text{surface term}}{=} 0$$

For complex scalar field ϕ we had

$$j^\mu = \phi\,\partial^\mu \phi^* - \phi^* \partial^\mu \phi \Rightarrow Q = \int d^3x \left[\phi\,\partial^0\phi^* - \phi^*\partial^0\phi\right].$$

For real scalar field we had $T^\mu{}_\nu$ which was conserved: $\partial_\mu T^{\mu\nu} = 0$.

$$\Rightarrow Q^\nu = \int d^3x\, T^{0\nu} \quad \sim 4 \text{ conserved charges}$$
$$\nu = 0, 1, 2, 3$$

$$\Rightarrow Q^0 = \int d^3x\, T^{00} = \int d^3x \left[\frac{\delta\mathcal{L}}{\delta(\partial_0\phi)}\,\partial^0\phi - \mathcal{L}\right]$$

In classical mechanics one had a <u>Hamiltonian</u>:

$$H = \sum_i p_i \dot{q}_i - L \Rightarrow \text{ we had } p_i = \frac{\delta L}{\delta \dot{q}^i}$$

$$\Rightarrow H = \sum_i \frac{\delta L}{\delta \dot{q}^i}\,\dot{q}^i - L.$$

The field theory analogue is (remember $L \to \int d^3x\,\mathcal{L}$)

$$\dot{\varphi} = \partial_0\varphi$$

$$H \equiv \int d^3x \left[\frac{\delta\mathcal{L}}{\delta\dot{\varphi}}\,\dot{\varphi} - \mathcal{L}\right] \equiv \int d^3x\, \mathcal{H}$$

$\Rightarrow$ we see that

$$Q^0 = \int d^3x \, T^{00} = \int d^3x \, \mathcal{H} = H$$ $\Rightarrow$ this is the

Hamiltonian! It is conserved: time translations lead to energy conservation!

$$Q^i = \int d^3x \, T^{0i} = \int d^3x \left[\frac{\delta \mathcal{L}}{\delta \dot{\phi}} \partial^i \phi \right]$$ $\Rightarrow$ interpret

as 3-momentum of the field.

for $m \neq 0$: $\mathcal{H} = (\partial^0 \varphi)^2 - \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{m^2}{2}\varphi^2 = \frac{1}{2}(\partial^0\varphi)^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2 \geq 0.$

$\Rightarrow \mathcal{H} \geq 0 \Rightarrow$ energy of the field ≥ 0 (not of a particle).

Lorentz & Poincare Groups and Classification of Fields

Before we start quantizing the fields, let us see what kinds of fields exist.

This can be accomplished by studying the group of Lorentz transformations.

We start by reviewing some group theory.

Def. Canonical momentum field

$$\boxed{\pi(x) \equiv \frac{\delta \mathcal{L}}{\delta \dot{\varphi}(x)}}$$

$\Rightarrow$ $\boxed{\mathcal{H} = \pi(x)\,\dot{\varphi}(x) - \mathcal{L}}$ Hamiltonian density

Real scalar field: $\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2}\varphi^2$

$$\pi(x) = \frac{\delta \mathcal{L}}{\delta \dot{\varphi}} = \dot{\varphi} \Rightarrow \mathcal{H} = \pi \underbrace{\dot{\varphi}}_{= \pi} - \mathcal{L} = \pi^2 - \frac{1}{2}\pi^2 +$$

$$+ \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2 \Rightarrow$$

$$\boxed{\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2}$$

$\mathcal{H} \geq 0 \Rightarrow$ energy of the <u>field</u> is non-negative.