$\Rightarrow$ we see that

$$Q^0 = \int d^3x \, T^{00} = \int d^3x \, \mathcal{H} = H$$ $\Rightarrow$ this is the

<u>Hamiltonian</u>! It is conserved: time translations lead to energy conservation!

$$Q^i = \int d^3x \, T^{0i} = \int d^3x \left[\frac{\delta \mathcal{L}}{\delta \dot{\phi}} \, \partial^i \phi \right]$$ $\Rightarrow$ interpret

as 3-momentum of the field.

for $m \neq 0$: $\mathcal{H} = (\partial^0 \varphi)^2 - \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{m^2}{2}\varphi^2 = \frac{1}{2}(\partial^0\varphi)^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2 \geq 0$.

$\Rightarrow \mathcal{H} \geq 0 \Rightarrow$ energy of the field ≥ 0 (not of a particle).

Lorentz & Poincare Groups and Classification of Fields

Before we start quantizing the fields, let us see what kinds of fields exist.

This can be accomplished by studying the group of Lorentz transformations.

We start by reviewing some group theory.

Elements of Group Theory

Def. A Group G is a set of elements with a multiplication law having the following properties:

(i) Closure: if $f, g \in G \Rightarrow h = f \cdot g \in G$

(ii) Associativity: $f, g, h \in G \Rightarrow f \cdot (g \cdot h) = (f \cdot g) \cdot h$

(iii) Identity: $\exists e \in G \ \forall f \in G : ef = fe = f$

(iv) Inverse element: $\forall f \in G \ \exists f^{-1} \in G : ff^{-1} = f^{-1}f = e$.

Example: $\{1, e^{i\frac{\pi}{2}}, e^{i\pi}, e^{i\frac{3}{2}\pi}\}$ form a group (why?). $\mathbb{Z}_4$ " $\{1, i, -1, -i\}$.

Integers: $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ form a group.

What is e there? Def. $H \subset G \Rightarrow H$ is a subgroup.

What is "multiplication" law?

Def. A group is called Abelian if for any

$f, g \in G$: $f \cdot g = g \cdot f$

otherwise it is called non-Abelian ($f \cdot g \neq g \cdot f$)

Example (important!) $n \times n$ unitary matrices form a group: $UU^+ = U^+U = \mathbb{1}$ (unitary matrices).

Def. Such group is denoted $U(n)$, $\left(e = \mathbb{1} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\right)$
and is called the unitary group.

Sub-example $U(1)$: 1×1 matrices $\Rightarrow e^{i\varphi}$, $\varphi \sim \#$

$\varphi \in \mathbb{R}$ – form a group, $e = 1$.

Def. $n \times n$ unitary matrices with unit determinant ($UU^+ = U^+U = \mathbb{1}$, $\det U = +1$) form a group too!

It is called special unitary group and is denoted $SU(n)$. (Orthogonal matrices $U^T U = U U^T = \mathbb{1}$ with $\det U = +1$ form $SO(n)$, O = orthogonal)

Def. A representation of group G is a mapping D of group elements : $f \in G$: $f \to D(f)$, where $D(f)$ is a space of linear operators (e.g. matrices) such that:

(i) $D(e) = \mathbb{1}$

(ii) $D(g_1) D(g_2) = D(g_1 g_2)$ for $g_1, g_2 \in G$.

Take a group $\mathbb{Z}_4$: it has $\{e, g_1, g_2, g_3\}$

Our example $\{1, e^{i\frac{\pi}{2}}, e^{i\pi}, e^{i\frac{3\pi}{2}}\} = \{D(e), D(g_1), D(g_2), D(g_3)\}$

is one of the many possible representations of $\mathbb{Z}_4$.

Def. Dimension of representation is the dimension of the space of D-matrices.

Def. Representation is called reducible if

$\exists M$ (matrix) such that

$$M\, D(g)\, M^{-1} = \begin{bmatrix} D_1(g) & 0 \\ 0 & D_2(g) \\ & & \ddots \end{bmatrix} \quad \text{for } \forall g \in G.$$

$\Rightarrow D = D_1 \oplus D_2 \oplus \ldots$

a representation is called irreducible if no such matrix M exists.

Def. For two groups $G = \{g_1, g_2, \ldots\}$, $H = \{h_1, h_2, \ldots\}$ define direct-product group $G \times H = \{g_i h_j\}$

such that $g_k h_e \cdot g_m h_n = g_k g_m \cdot h_e h_n$.

Lie Groups

Imagine a group G with elements smoothly dependent on a continuous set of parameters α_i, $i = 1, \ldots, N$: $g(\alpha_i) \in G$.

$\Rightarrow$ assume that $g(\alpha_i = 0) = e$ (the identity element)

$\Rightarrow$ for a representation of the group:

$$D(\alpha_i = 0) = \mathbb{1}.$$

Taylor expand $D(\alpha_i)$ near 0:

$$D(\delta\alpha_i) = \mathbb{1} + i\,\delta\alpha_i X_i + \ldots = \mathbb{1} + i\,\delta\vec{\alpha}\cdot\vec{X}$$

(summation over repeating indices assumed)

Def. X_i are called generators of the group.

$\delta\vec{\alpha} = \frac{\vec{\alpha}}{k}$, k ~ integer

$$D(\alpha_i) = D(\delta\alpha_i)\,D(\delta\alpha_i)\ldots D(\delta\alpha_i) = \lim_{k\to\infty}\left(\mathbb{1} + i\,\delta\vec{\alpha}\cdot\vec{X}\right)^k$$

$$= \lim_{k\to\infty}\left(\mathbb{1} + i\,\frac{\vec{\alpha}}{k}\cdot\vec{X}\right)^k = e^{i\vec{\alpha}\cdot\vec{X}}.$$

Def. A group with elements depending smoothly on continuous set of parameters α_i, $i=1,\ldots,N$, with generators X_i is called a Lie group.

$D(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{X}}$ $\Rightarrow$ as D can be a matrix

$\Rightarrow$ $\vec{X}$ can be a matrix; therefore in general $[X_i, X_j]$ does not have to be 0.

$\parallel$ $X_i X_j - X_j X_i$