<u>Last time</u>: Applied Noether's theorem to space-time translations symmetry (btw, N. th'm applies for continuous symmetries only):

$x^\mu \to x^\mu - \delta a^\mu \Rightarrow \phi(x) \to \phi(x'^\mu) + \delta a^\mu \partial'_\mu \phi(x')$

$\Rightarrow$ obtained a conserved current

$$\boxed{T_{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial^\mu \phi)} \partial_\nu \phi - g_{\mu\nu} \mathcal{L}}$$ energy-momentum tensor

$\boxed{\partial_\mu T^{\mu\nu} = 0}$ ~ conserved tensor

<u>Defined</u> Hamiltonian $\boxed{H = \int d^3x \left[\pi \cdot \dot{\varphi} - \mathcal{L}\right]}$

with $\boxed{\pi = \frac{\delta \mathcal{L}}{\delta \dot{\varphi}}}$ the canonical momentum

conserved charges $Q^\nu = \int d^3x\, T^{0\nu} \Rightarrow Q^0$ is the Hamiltonian $Q^0 = H$, Q^i ~ momentum of the field

Lorentz & Poincare Groups and Classification of Fields (cont'd)

Elements of Group Theory (cont'd)

Group G: (i) $f \cdot g \in G$ (ii) $f \cdot (g \cdot h) = (f \cdot g) \cdot h$ (iii) $\exists e : f \cdot e = e \cdot f = f$

$f, g, h \in G$ (iv) $\forall f\ \exists f^{-1} : f^{-1} f = f \cdot f^{-1} = e$

$f\cdot g = g\cdot f$ Abelian $\quad f\cdot g \neq g\cdot f$ non-Abelian

Def. group of unitary $n\times n$ matrices, $UU^\dagger = U^\dagger U = \mathbb{1}$, along with $\det U = +1$ = special unitary group, $SU(n)$

Def. -"- orthogonal -"- $\quad UU^T = U^T U = \mathbb{1} \Rightarrow SO(n)$

Def. <u>Group representation</u>: $f \in G$: $f \to D(f)$ ← repr.

(i) $D(e) = \mathbb{1}$ $\quad$ (ii) $D(g_1)\, D(g_2) = D(g_1 g_2)$, $\forall g_1, g_2 \in G$.

dimension of representation = dim of $D(g)$

Def. Reducible representation: $\exists M$:

$$M\, D(g)\, M^{-1} = \begin{bmatrix} D_1(g) & 0 & 0 \\ 0 & D_2(g) & 0 \\ 0 & 0 & \ddots \end{bmatrix} \Rightarrow D = D_1 \oplus D_2 + \ldots$$

no such M ⇒ irreducible representation

Def. Direct-product $G \otimes H = \{g_i\, h_j\} \Rightarrow g_k h_e \cdot g_m h_n = $
$= g_k \cdot g_m\, h_e \cdot h_n$

Lie Groups:

Def. Lie group,
$g(\alpha^i)$, $i = 1, \ldots, N$, α^i ~ continuous set of parameters

can represent group elements as $\left(\vec{\alpha} = (\alpha^1, \alpha^2, \ldots, \alpha^N)\right)$

$D(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{X}}$, X_i ~ generators of Lie group.

$\Rightarrow$ assume that $g(\alpha_i = 0) = e$ (the identity element)

$\Rightarrow$ for a representation of the group:

$$D(\alpha_i = 0) = \mathbb{1}.$$

Taylor expand $D(\alpha_i)$ near 0:

$$D(\delta\alpha_i) = \mathbb{1} + i\,\delta\alpha_i X_i + \dots = \mathbb{1} + i\,\delta\vec{\alpha}\cdot\vec{X}$$

(summation over repeating indices assumed)

Def.
X_i are called <u>generators</u> of the group.

$\delta\vec{\alpha} = \frac{\vec{\alpha}}{k}$, k ~ integer

$$\Rightarrow D(\alpha_i) = D(\delta\alpha_i)\,D(\delta\alpha_i)\dots D(\delta\alpha_i) = \lim_{k\to\infty}\left(\mathbb{1} + i\,\delta\vec{\alpha}\cdot\vec{X}\right)^k$$

$$= \lim_{k\to\infty}\left(\mathbb{1} + i\,\frac{\vec{\alpha}}{k}\cdot\vec{X}\right)^k = e^{i\vec{\alpha}\cdot\vec{X}}.$$

Def. A group with elements depending smoothly on continuous set of parameters α_i, $i = 1, \dots, N$, with generators X_i is called a <u>Lie group</u>.

$D(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{X}}$ $\Rightarrow$ as D can be a matrix

$\Rightarrow$ $\vec{X}$ can be a matrix; therefore in general $[X_i, X_j]$ does not have to be 0.

$$[X_i, X_j] = X_i X_j - X_j X_i$$

$\Rightarrow$ however $D(\vec{\alpha})\, D(\vec{\beta}) = e^{i\vec{\alpha}\cdot X}\, e^{i\vec{\beta}\cdot X}$ is

also a group element $\Rightarrow e^{i\vec{\alpha}\cdot\vec{X}}\, e^{i\vec{\beta}\cdot\vec{X}} = e^{i\vec{\delta}\cdot\vec{X}}$

$\Rightarrow$ can show that for this to work we need

$$[X_a, X_b] = i f_{abc} X_c$$ Lie algebra of generators

f_{abc} ~ structure constants of the group

$f_{abc} = - f_{bac}$;

Def. Commutator:

$$[A,B] = A\cdot B - B\cdot A.$$

f_{abc} are <u>real</u> for unitary representations D

(for hermitean X_a): $D^+ D = D D^+ = \mathbb{1}$

<u>Example</u> take the group $SU(2)$: unitary 2×2 matrices with det $= +1$ $(uu^+ = u^+u = \mathbb{1}, \det u = +1)$.
(defining representation)
Using Pauli matrices we can define a representation of $SU(2)$:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\Rightarrow D(\vec{\alpha}) = e^{i\frac{\vec{\alpha}\cdot\vec{\sigma}}{2}}$, $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ a 3-vector.

rotations around $\frac{\vec{\alpha}}{|\vec{\alpha}|}$ axis by angle $|\vec{\alpha}|$.

as $\sigma_i^+ = \sigma_i$ (hermitean) $\Rightarrow$ any 2×2 unitary matrix with det = +1 can be represented as $e^{i\frac{\vec{\alpha}\cdot\vec{\sigma}}{2}} = U$.

Check: $UU^+ = e^{i\frac{\vec{\alpha}\cdot\vec{\sigma}}{2}} e^{-i\frac{\vec{\alpha}\cdot\vec{\sigma}}{2}} = \mathbb{1}$.

$\det U = \det e^{i\frac{\vec{\alpha}\cdot\vec{\sigma}}{2}} = \left[\text{as } \det e^A = e^{\operatorname{tr} A}\right] = 1$

as $\operatorname{tr}\sigma_i = 0$. $\overset{\text{comp's}}{8} - \overset{\text{cond's}}{4} = 4$ (linearly independent)

$\Rightarrow$ there are $2^2 - 1 = 3$ different $n\times n$ traceless hermitean matrices $\Rightarrow \{\sigma_i\}$ use up all possibilities.

Generators: $J_i = \frac{\sigma_i}{2} \Rightarrow D(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{J}}$.

$\Rightarrow SU(2)$ is a Lie group

We know that $[\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\sigma_k \Rightarrow$

$$\Rightarrow [J_i, J_j] = i\,\varepsilon_{ijk} J_k$$

$\Rightarrow$ generators of $SU(2)$ form a Lie algebra with structure constants ε_{ijk}

ε_{ijk} : totally anti-symmetric Levi-Civita symbol, $\varepsilon_{123} = 1$, $\varepsilon_{ijk} = -\varepsilon_{jik} = \varepsilon_{jki}$ ...
$\varepsilon_{112} = 0$ ....

Another example: SU(3) : 3×3 unitary matrices with $\det = +1$

Define Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ (cf. Pauli matrices)

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Normalization convention $\mathrm{tr}[\lambda_i \lambda_j] = 2\,\delta_{ij}$.

There are $3^2 - 1 = 8$ traceless hermitean 3×3 matrices

$\Rightarrow$ these should work.

Generators of SU(3): $T^a = \dfrac{\lambda^a}{2}$ $\Rightarrow$

$\Rightarrow$ $[T^a, T^b] = i f^{abc}\, T^c$, with structure constants f^{abc}, which are anti-symmetric under the interchange of any two indeces.

$\Rightarrow$ SU(3) is a Lie group with the generator algebra given above.

a	b	c	f^{abc}
1	2	3	1
1	4	7	1/2
1	5	6	-1/2
2	4	6	1/2
2	5	7	1/2
3	4	5	1/2
3	6	7	-1/2
4	5	8	$\sqrt{3}/2$
6	7	8	$\sqrt{3}/2$

$f_{112} = 0 \ \ldots$

all other f^{abc}'s can be obtained from this table.

Casimir operator commutes with all generators:

$$\vec{T}^2 = T_1^2 + T_2^2 + \ldots + T_n^2 = \frac{N^2-1}{2N}$$

$\Rightarrow$ for SU(2) it is $3/4$

for SU(3) it is $4/3$.

$$D(\vec{A}) = e^{i\vec{A}\cdot\vec{T}}, \text{ with } \vec{A} = (A_1, A_2, \ldots, A_8)$$

~ an 8-component vector.

Jacobi Identity and the Adjoint Representation

~ go back to some general Lie group with the generators X_a obeying some Lie algebra

$$[X_a, X_b] = i f_{abc} X_c .$$

One can then easily prove Jacobi identity:

$$[X_a,[X_b,X_c]]+[X_b,[X_c,X_a]]+[X_c,[X_a,X_b]]=0.$$

(prove this by using definitions of commutators)

$\Rightarrow$ plug in the commutator of Lie algebra to write

$$f_{bdc}f_{ade}+f_{abd}f_{cde}+f_{cad}f_{bde}=0$$

this relations are obeyed by structure constants of any Lie group, e.g. $SU(n)$.

Define The generators in the adjoint representati[on]

by $(t^a)_{bc}=-i f_{abc}$ $\Rightarrow$ the above relation

gives $[t^a,t^b]=i f_{abc}\,t^c$.

$\Rightarrow$ they obey the Lie algebra too!

Def. $D(\vec{A})=e^{iA^a t^a}$ gives the <u>adjoint</u> representation of Lie group.

Lorentz Group

Work in Minkowski space, $\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

$\eta_{\mu\nu}\eta^{\nu\rho} = \delta_\mu{}^\rho$; $\quad x_\mu = \eta_{\mu\nu} x^\nu = (t, -\vec{x})$, $x^\mu = (t, \vec{x})$.

Def. Set of linear (real) transformations

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

forms the Lorentz group if

$$\eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} x^\mu x^\nu$$

(proper time is preserved). $\qquad \eta_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$ metric tensor

---

Example: $\Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ for boosts along x^1-axis.

$$\eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} x^\mu x^\nu$$

$$\eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta x^\alpha x^\beta = \eta_{\alpha\beta} x^\alpha x^\beta$$

$$\Rightarrow \boxed{\eta_{\alpha\beta} = \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta}$$

or, equivalently, $\eta = \Lambda^T \eta \Lambda$

as $\eta_{\mu\nu}\eta^{\nu\rho} = \delta_\mu{}^\rho \Rightarrow \eta \cdot \eta = \mathbb{1}$

$$\Rightarrow \quad \eta \cdot \eta = \mathbb{1} = \eta \Lambda^T \eta \Lambda \quad \Rightarrow \quad \boxed{\Lambda^{-1} = \eta \Lambda^T \eta}$$

$$\Rightarrow \quad \eta = \Lambda \eta \Lambda^T$$ (multiply by $\Lambda \Rightarrow \mathbb{1} = \Lambda \eta \Lambda^T \eta$) & by η on the right.

Why is this set a group? Λ_1, Λ_2 ~ different $\mathcal{L}$. tr.

(i) $\Lambda = \Lambda_2 \cdot \Lambda_1 \quad \Rightarrow \quad \Lambda^\mu{}_\nu = \Lambda_2{}^\mu{}_\rho \Lambda_1{}^\rho{}_\nu$

$$\eta = \Lambda_1^T \eta \Lambda_1 \, , \quad \eta = \Lambda_2^T \eta \Lambda_2$$

$$\Rightarrow \quad (\Lambda_2 \Lambda_1)^T \eta (\Lambda_2 \Lambda_1) = \Lambda_1^T \underbrace{\Lambda_2^T \eta \Lambda_2}_{\eta} \Lambda_1 = \Lambda_1^T \eta \Lambda_1 = \eta$$

$\Rightarrow \quad \Lambda \in$ Lorentz group

(ii) $\Lambda_1 \cdot (\Lambda_2 \cdot \Lambda_3) = (\Lambda_1 \cdot \Lambda_2) \cdot \Lambda_3$ trivially true for matrices

(iii) Identity: $\delta^\mu{}_\nu = \mathbb{1}$ exists.

(iv) $\forall \, \Lambda \in$ Lorentz group there exists

$$\Lambda^{-1} = \eta \Lambda^T \eta \; : \quad \Lambda \Lambda^{-1} = \Lambda^{-1} \Lambda = \mathbb{1}.$$

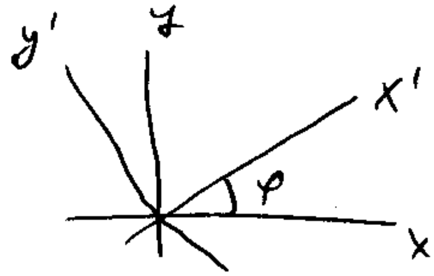
$\Rightarrow$ Lorentz group is a group.

Examples of Lorentz group elements:

(1) Usual Lorentz transformation:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(2) Rotation in x-y plane:



$$x \to x' = x\cos\varphi + y\sin\varphi$$

$$y \to y' = -x\sin\varphi + y\cos\varphi$$

$$= \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \quad \Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(3) Parity: $\vec{x} \to -\vec{x}$,
$\mathbb{P}$

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

(4) Time reversal, $\mathbb{T}$: $t \to -t$,

$$\Lambda^\mu{}_\nu = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}.$$