<u>Last time</u>: finished talking about Lie groups:

$$D(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{X}}, \quad [X^a, X^b] = i f^{abc} X^c \quad \text{generator algebra}$$

$\uparrow$ structure const's.

Considered examples of

<u>SU(2)</u> $D(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}}$, $\vec{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$, $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$

σ^i ~ Pauli matrices. Generators: $J^i = \frac{\sigma^i}{2}$ with the algebra $[J^i, J^j] = i\varepsilon^{ijk} J^k$

$\varepsilon^{123} = +1$, $\varepsilon^{213} = -1$, $\varepsilon^{113} = 0$, ... Levi-Civita symbol

<u>SU(3)</u> $D(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{T}}$, $\vec{\alpha} = (\alpha^1, \dots, \alpha^8)$, $\vec{T} = \frac{\vec{\lambda}}{2}$,

λ^i ~ Gell-Mann matrices, $[T^a, T^b] = i f^{abc} T^c$

$\uparrow$ SU(3) structure constants (real)

Def.

$(t^a)_{bc} \equiv -i f^{abc}$ ~ give adjoint representation (real matrices)

Degrees of freedom: $\underbrace{3\times3\times2}_{\text{size of } T\text{'s}} \underbrace{\,}_{\text{in c.c.}} - \underbrace{3\times3}_{u^+u=1} - \underbrace{1}_{\det u=1} = 8.$

Dimension of adjoint representation = # d. of f. in the group.

Lorentz Group (cont'd):

Def. real $\Lambda^\mu{}_\nu$'s : $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ such that

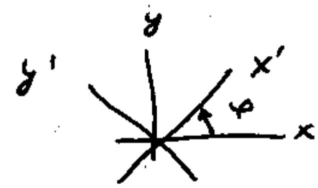
$$\eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} x^\mu x^\nu$$

In matrix notation: $\eta = \Lambda^T \eta \Lambda$.

Labeled : $SO(3,1)$ to signify that time is different.

<u>Examples</u>: (1) boost $\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(2) Rotation $\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$



(3) Parity $\mathbb{P}$: $\vec{x} \to -\vec{x}$

(4) Time reversal $\mathcal{T}$: $t \to -t$.

Now, $\eta = \Lambda^T \eta \Lambda \Rightarrow \cancel{\det \eta} = \underbrace{\det \Lambda^T}_{= \det \Lambda} \cancel{\det \eta} \det \Lambda$

$\Rightarrow$ $\det \Lambda = \pm 1$. "+" proper, "−" improper LT's.

also, $1 = \eta_{00} = \Lambda^\mu{}_0 \Lambda^\nu{}_0 \eta_{\mu\nu} = \Lambda^0{}_0 \Lambda^0{}_0 - \Lambda^i{}_0 \Lambda^i{}_0$

$\Rightarrow 1 = (\Lambda^0{}_0)^2 - (\Lambda^i{}_0)^2 \Rightarrow |\Lambda^0{}_0| \geq 1 \Rightarrow$

$\Rightarrow$ either $\boxed{\Lambda^0{}_0 \geq 1}$ or $\boxed{\Lambda^0{}_0 \leq -1}$.

orthochronous   non-orthocronous

4 types of transformations:

$\det \Lambda = +1$, $\Lambda^0{}_0 \geq 1$ (e.g. boosts, rotations)

$\det \Lambda = +1$, $\Lambda^0{}_0 \leq -1$ (e.g. full inversion $x^\mu \to -x^\mu$)

$\det \Lambda = -1$, $\Lambda^0{}_0 \geq 1$ (parity $\mathbb{P}$)

$\det \Lambda = -1$, $\Lambda^0{}_0 \leq -1$ (time reversal $\mathbb{T}$).

Representations of Lorentz group.

Consider infinitesimal Lorentz transformation:

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \Rightarrow \eta^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta^{\alpha\beta}$$

$\uparrow$ small

$$\eta^{\mu\nu} = \left(\delta^\mu{}_\alpha + \omega^\mu{}_\alpha\right)\left(\delta^\nu{}_\beta + \omega^\nu{}_\beta\right)\eta^{\alpha\beta} =$$

$$= \eta^{\mu\nu} + \omega^{\mu\nu} + \omega^{\nu\mu} + O(\omega^2)$$

$$\Rightarrow \boxed{\omega^{\mu\nu} + \omega^{\nu\mu} = 0} \quad \Rightarrow \omega^{\mu\nu} = -\omega^{\nu\mu}$$

anti-symmetric real tensor, 4×4 matrix

⇒ has 6 independent components.

⇒ L. group should have 6 generators. → 3 boosts, ↳ 3 rotations

Want to write L. transformations as

$$\Lambda = e^{\frac{i}{2}\omega^{\mu\nu}L_{\mu\nu}}$$

with the alebra of generators $L_{\mu\nu}$.

Def. Generators of L. group

$$\boxed{L_{\mu\nu} \equiv i\left[x_\mu \partial_\nu - x_\nu \partial_\mu\right]}$$

⇒ let's check that these work:

$$x^\mu \to x'^\mu = e^{\frac{i}{2}\omega^{\alpha\beta}L_{\alpha\beta}}\, x^\mu = \left(1 + \frac{i}{2}\omega^{\alpha\beta}L_{\alpha\beta}\right)x^\mu$$

$$= x^\mu + \frac{i}{2}\cdot i\,\omega^{\alpha\beta}\left(x_\alpha\partial_\beta - x_\beta\partial_\alpha\right)x^\mu =$$

$$= x^\mu - \omega^{\alpha\beta}\,x_\alpha\partial_\beta x^\mu = x^\mu - \omega^{\alpha\beta}\,x_\alpha\,\delta^\mu_\beta =$$

$$= x^\mu - \omega^{\alpha\mu} x_\alpha = x^\mu + \omega^{\mu\alpha} x_\alpha =$$

$$= \left(\delta^\mu{}_\alpha + \omega^\mu{}_\alpha \right) x^\alpha \Rightarrow \text{works!}$$

Lie algebra of L. group generators is

$$\boxed{[L_{\mu\nu}, L_{\rho\sigma}] = i\eta_{\nu\rho} L_{\mu\sigma} - i\eta_{\mu\rho} L_{\nu\sigma} - i\eta_{\nu\sigma} L_{\mu\rho} + i\eta_{\mu\sigma} L_{\nu\rho}}$$

Lorentz group is denoted by $SO(3,1)$.

Consider L_{ij} first. <u>Define</u> $L_i \equiv \frac{1}{2} \varepsilon_{ijk} L_{jk}$

$(\varepsilon_{123} = 1, \quad \varepsilon_{iik} = \dots = 0, \quad \varepsilon_{213} = -1, \dots)$

Levi-Civita symbol.

$$L_1 = \frac{1}{2} \left[\overset{=+1}{\varepsilon_{123}} L_{23} + \overset{=-1}{\varepsilon_{132}} L_{32} \right] = \frac{1}{2} \left(L_{23} - L_{32} \right) = L_{23}$$

$$= -i \left(y \partial_z - z \partial_y \right) = -i \left(\vec{x} \times \vec{\nabla} \right)_1$$

as $y = -x_2 = x^2$, $z = -x_3 = x^3$

$\Rightarrow$ in general $\vec{L} = -i \, \vec{x} \times \vec{\nabla}$

In QM momentum operator is $\vec{p} = -i \vec{\nabla}$

$\Rightarrow$ $\vec{L} = \vec{X} \times \vec{p}$ ~ orbital angular momentum

$\Rightarrow$ All L_{ij} generators, $i,j=1,2,3$, combine to give $\vec{L}$.

$\Rightarrow$ What do $\vec{L}$ generate?

$$x^\mu \to x'^\mu = e^{\frac{i}{2}\omega^{ij} L_{ij}} x^\mu$$

$$L_{ij} \overset{\text{claim}}{=} \varepsilon_{ijk} L_k \overset{\text{check}}{=} \varepsilon_{ijk} \frac{1}{2} \varepsilon_{klm} L_{lm} = \frac{1}{2}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) L_{lm} = \frac{1}{2}(L_{ij} - L_{ji}) = L_{ij} \quad \text{works}$$

$$\Rightarrow x'^\mu = e^{\frac{i}{2}\omega^{ij}\varepsilon_{ijk} L_k} x^\mu$$

As $\omega_{ij} = -\omega_{ji}$ $\Rightarrow$ define $\theta_i = \frac{1}{2}\varepsilon_{ijk}\omega_{jk}$

$$\Rightarrow x'^\mu = e^{i\vec{\theta}\cdot\vec{L}} x^\mu$$

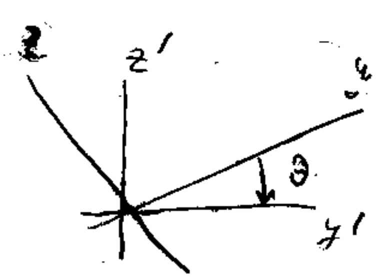
For example take $\vec{\theta} = (\overset{=\theta_1}{\theta}, 0, 0) \Rightarrow e^{i\overbrace{\vec{\theta}\cdot\vec{L}}^{=\theta_1 L_1}} x^\mu =$

$$= e^{+i\theta L_1} x^\mu = e^{\theta(y\partial_z - z\partial_y)} x^\mu$$

$\Rightarrow$ $t' = t$, $x' = x$;

if θ ~ tiny $\Rightarrow$ $\delta x^\mu = \theta(y\partial_z - z\partial_y) x^\mu$

$\Rightarrow$ $\delta y = -\theta z$; $\delta z = \theta y$



$$\begin{cases} y' = y - \theta z \\ z' = z + \theta y \end{cases}$$ rotation by θ in the clockwise direction
(following right-hand rule)

$\Rightarrow$ L_{ij} and therefore $\vec{L}$ generate rotations!

What about the remaining generators L_{0i}?

Def. $\boxed{K_i = L_{0i}}$ (transforms as 3-vector under spatial rotations; not so under boosts)

$$x'^\mu = e^{\frac{i}{2}\omega^{\mu\nu}L_{\mu\nu}} x^\mu = e^{\frac{i}{2}\cdot 2\cdot \omega^{0i}L_{0i}} x^\mu =$$

$$= e^{i\omega^{0i}K_i} x^\mu$$

$\Rightarrow$ take $\xi^i = \omega^{0i}$ $\Rightarrow$ $x'^\mu = e^{-i\vec{\xi}\cdot\vec{K}} x^\mu$

Take $\vec{\xi} = (\xi^1, 0, 0)$ $\Rightarrow$ $x'^\mu = e^{-i\xi^1 K^1} x^\mu =$

$$= e^{-i\xi^1\cdot i\cdot[x^0\partial^1 - x^1\partial^0]} x^\mu = e^{-\xi^1[x^0\partial_1 + x^1\partial_0]} x^\mu$$

$$= e^{-\xi^1[t\partial_x + x\partial_t]} x^\mu$$ $\Rightarrow$ assume that $\xi_1 \ll 1$ $\Rightarrow$

$\Rightarrow$ $y' = y$, $z' = z$

$t' = t - \xi^1 x$; $x' = x - \xi^1 t$

$$\Rightarrow \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & -\xi^1 & 0 & 0 \\ -\xi^1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$ ~ boost along x-axis with $\beta = \xi^1 \ll 1$.