Last time: we showed that an element of the (restricted) Lorentz group can be represented by:

$$\boxed{U(\Lambda) = e^{\frac{i}{2}\omega^{\mu\nu} L_{\mu\nu}}}$$

where $L_{\mu\nu} = i\left[X_\mu \partial_\nu - X_\nu \partial_\mu\right]$ are the generators of Lorentz group and $\omega^{\mu\nu} = -\omega^{\nu\mu}$ is an anti-symmetric tensor.

Generator algebra

$$[L_{\mu\nu}, L_{\rho\sigma}] = i\eta_{\nu\rho} L_{\mu\sigma} - i\eta_{\mu\rho} L_{\nu\sigma} - i\eta_{\nu\sigma} L_{\mu\rho} + i\eta_{\mu\sigma} L_{\nu\rho}$$

Defined $L_i = \frac{1}{2}\varepsilon_{ijk} L_{jk}$ & showed that

$\vec{L} = -i\,\vec{x} \times \vec{\nabla} = \vec{x} \times \vec{p}$ is the angular momentum operator in QM.

Showed that $e^{\frac{i}{2}\omega^{ij} L_{ij}} = e^{i\vec{\theta}\cdot\vec{L}}$ generates rotations by angle $|\vec{\theta}|$ around $\vec{\theta}$ directions according to the right-hand rule.

Note on sign convention: $A_i = -A^i$ for $i = 1, 2, 3$.

$A_{ij} = A^{ij}(-1)^2 = A^{ij}$; $A_{ijk} = (-1)^3 A^{ijk} = -A^{ijk}$

ε_{ijk} : $\varepsilon_{123} = 1$. What is ε^{123} ?

ε_{ijk} is <u>not</u> a Lorentz-tensor, hence it makes no sense to raise/lower its indices, can't put $\varepsilon^{123} = -1$, would be a mess $\Rightarrow$ agree that $\varepsilon_{ijk} = \varepsilon^{ijk}$.

Def. $\theta_i = \frac{1}{2} \varepsilon_{ijk} \omega_{jk}$

$$\Rightarrow e^{\frac{i}{2}\omega^{ij} L_{ij}} \underset{L_{ij} = \varepsilon_{ijk} L_K}{=} e^{\frac{i}{2}\omega^{ij}\varepsilon_{ijk} L_K} = e^{\frac{i}{2}\varepsilon_{ijk}\omega_{ij} L_K}$$

$$= e^{i\theta_k L_K} \qquad = e^{i\vec{\theta}\cdot\vec{L}}$$

Take $\vec{\theta} = (\underset{\theta^1}{\theta}, \underset{\theta^2}{0}, \underset{\theta^3}{0}) \Rightarrow e^{i\vec{\theta}\cdot\vec{L}} = e^{i\theta \cdot L^1}$

$$L_1 = L_{23} = i(x_2\partial_3 - x_3\partial_2) = -i(x^2\partial_3 - x^3\partial_2)$$

$$= -i(y\partial_z - z\partial_y)$$

$$\Rightarrow e^{i\vec{\theta}\cdot\vec{L}} = e^{\theta(y\partial_z - z\partial_y)}$$ $\Rightarrow$ clockwise rotation.

$$\begin{cases} y' = y - \theta z \\ z' = z + \theta y \end{cases}$$ rotation by θ in the clockwise direction (following right-hand rule)

$\Rightarrow$ L_{ij} and therefore $\vec{L}$ generate <u>rotations</u>!

What about the remaining generators L_{0i}?

Def. $\boxed{K_i = L_{0i}}$ (transforms as 3-vector under spatial rotations; not so under boosts)

$$x'^\mu = e^{\frac{i}{2}\omega^{\mu\nu}L_{\mu\nu}}\, x^\mu = e^{\frac{i}{2}\cdot 2\cdot \omega^{0i}L_{0i}}\, x^\mu =$$

$$= e^{i\omega^{0i}K_i}\, x^\mu$$

$\Rightarrow$ take $\xi^i = \omega^{0i}$ $\Rightarrow$ $x'^\mu = e^{-i\vec{\xi}\cdot\vec{K}}\, x^\mu$

Take $\vec{\xi} = (\xi^1, 0, 0)$ $\Rightarrow$ $x'^\mu = e^{-i\xi^1 K^1}\, x^\mu =$

$$= e^{-i\xi^1\cdot i\cdot[x^0\partial^1 - x^1\partial^0]}\, x^\mu = e^{-\xi^1[x^0\partial_1 + x^1\partial_0]}\, x^\mu$$

$$= e^{-\xi^1[t\,\partial_x + x\,\partial_t]}\, x^\mu$$ $\Rightarrow$ assume that $\xi_1 \ll 1$ $\Rightarrow$

$\Rightarrow$ $y' = y$, $z' = z$

$t' = t - \xi^1 x$; $x' = x - \xi^1 t$

$$\Rightarrow \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & -\xi^1 & 0 & 0 \\ -\xi^1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

~ boost along x-axis with $\beta = \xi^1 \ll 1$.

We have 2 types of generators:

$$\boxed{\begin{aligned} \vec{L} &= -i\, \vec{x} \times \vec{\nabla} \\ \vec{K} &= -i\left[x^0 \vec{\nabla} + \vec{x}\, \partial_0 \right] \end{aligned}}$$

3 rotations

3 boosts

$$e^{\frac{i}{2}\omega^{\mu\nu} L_{\mu\nu}} = e^{-i\vec{\xi}\cdot\vec{K} + i\vec{\theta}\cdot\vec{L}} = \Lambda$$

Consider a scalar field $\phi(x)$: under L. transform we have $\phi(x) \to \phi'(x') = \phi(x)$ does not change

$$\Rightarrow \phi'(x) = \phi(\Lambda^{-1} x) \quad \text{as} \quad x' = \Lambda \cdot x .$$

Define a representation of Lorentz transformation $U(\Lambda)$ on the space of fields $\phi(x)$:

$$\Lambda \to U(\Lambda)$$

by

$$\phi'(x) \equiv U(\Lambda)\, \phi(x) \equiv \phi(\Lambda^{-1} x)$$

To find $U(\Lambda)$ consider infinitesimal transform.

$$x^\mu \to \Lambda x^\mu = x^\mu + \delta x^\mu \Rightarrow \Lambda^{-1} x^\mu = x^\mu - \delta x^\mu$$

$$\Rightarrow U(\Lambda)\, \phi(x) = \phi(\Lambda^{-1} x^\mu) = \phi(x^\mu - \delta x^\mu) =$$

$$= \phi(x) - \delta x^\mu \partial_\mu \phi(x)$$

as $x'^\mu = e^{\frac{i}{2}\omega^{\alpha\beta} L_{\alpha\beta}} x^\mu \approx \left(1 + \frac{i}{2}\omega^{\alpha\beta} L_{\alpha\beta}\right) x^\mu$

$$= \left(\delta^\mu_\alpha + \omega^\mu{}_\alpha\right) x^\alpha = x^\mu + \omega^\mu{}_\alpha x^\alpha \Rightarrow \delta x^\mu = \omega^\mu{}_\alpha x^\alpha$$

$$\Rightarrow \quad U(\Lambda)\phi(x) = \phi(x) - \omega^\mu{}_\alpha x^\alpha \partial_\mu \phi(x) =$$

$$= \phi(x) - \omega^{\mu\alpha} \frac{1}{2} \underbrace{\left(x_\alpha \partial_\mu - x_\mu \partial_\alpha\right)}_{-i L_{\alpha\mu}} \phi(x)$$

$$= \phi(x) - \frac{i}{2} \omega^{\mu\nu} L_{\mu\nu} \phi(x) = \left(1 - \frac{i}{2}\omega^{\mu\nu} L_{\mu\nu}\right)\phi(x)$$

$$\Rightarrow \boxed{U(\Lambda) = e^{-\frac{i}{2}\omega^{\mu\nu} L_{\mu\nu}}} = \boxed{e^{i\vec{\xi}\cdot\vec{K} - i\vec{\theta}\cdot\vec{L}}}$$

One can show that

$$\begin{cases} [L_i, L_j] = i\varepsilon_{ijk} L_k \\ [L_i, K_j] = i\varepsilon_{ijk} K_k \\ [K_i, K_j] = -i\varepsilon_{ijk} L_k \end{cases}$$

another way of writing Lorentz <u>generator</u> algebra.

For particles with $\neq 0$ spin have the spin operator too, denoted $\vec{S}$. as you know from QM course $[S_i, S_j] = i\varepsilon_{ijk} S_k$.

Defining angular momentum

$$\boxed{\vec{J} = \vec{L} + \vec{S}}$$

we have
$$\begin{cases} [J_i, J_j] = i\varepsilon_{ijk} J_k \\ [J_i, K_j] = i\varepsilon_{ijk} K_k \\ [K_i, K_j] = -i\varepsilon_{ijk} J_k \end{cases}$$

if we define $\boxed{J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}}$

with $S_{\mu\nu}$ satisfying the same commutation relations as $L_{\mu\nu}$ and $[L_{\alpha\beta}, S_{\mu\nu}] = 0$.

Then
$$J_i = \frac{1}{2}\varepsilon_{ijk} J_{jk}$$
$$K_i = J_{0i} \qquad = L_{0i} + S_{0i}$$

and the above algebra is satisfied!

Note that $\vec{J}, \vec{K}$ are hermitean!

Define $\boxed{\vec{N}_+ \equiv \frac{1}{2}(\vec{J} + i\vec{K})}$ $\boxed{\vec{N}_- = \vec{N}_+^{\dagger}}$

but $[N_+^i, N_-^j] = 0$ $\qquad \vec{N}_- = \frac{1}{2}(\vec{J} - i\vec{K}^{\dagger})$

check: (w/o spin) $\left[\frac{1}{2}(J^i + iK^i), \frac{1}{2}(J^j - iK^j)\right] =$

$$= \frac{1}{4}\left\{ \cancel{i\varepsilon^{ijk} J^k} - \cancel{i\varepsilon^{ijk} J^k} + \frac{i}{2}[K^i, J^j] + \frac{i}{2}[K^j, J^i] \right\} = 0$$

Def. $\vec{N}_+ \equiv \frac{1}{2}\left[\vec{J} + i(\vec{K} - i\vec{S})\right] = \frac{1}{2}\left[\vec{J} + i\vec{K} + \vec{S}\right] = \frac{1}{2}\left[\vec{L} + i\vec{K}\right] + \vec{S}$

$$\vec{N}_- \overset{=\vec{N}_+^{\dagger}}{=} \frac{1}{2}\left[\vec{J} - i(\vec{K} + i\vec{S})\right] = \frac{1}{2}\left[\vec{L} + \vec{S} - i\vec{K} + \vec{S}\right]$$

$$= \frac{1}{2}\left[\vec{L} - i\vec{K}\right] + \vec{S}$$

in $\vec{N}_+$, the spin operator acts in left-handed space, in $\vec{N}_-$ it acts in right-handed space

$\Rightarrow$ the spins commute. $\Rightarrow$ still have $[N_+^i, N_-^j] = 0$.

Under parity: $\vec{L} \xrightarrow{\mathbb{P}} \vec{L}$, $\vec{K} \xrightarrow{\mathbb{P}} -\vec{K}$

$$\Rightarrow \begin{cases} \vec{N}_+ \xrightarrow{\mathbb{P}} \vec{N}_- \\ \vec{N}_- \xrightarrow{\mathbb{P}} \vec{N}_+ \end{cases} \sim \text{just swap.}$$

$$\left.\begin{aligned} \Lambda_L &= e^{i\vec{\xi}\cdot\left(-i\frac{\vec{\sigma}}{2}\right) - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}} = e^{-i\frac{\vec{\sigma}}{2}\cdot(\vec{\theta} + i\vec{\xi})} \\ \Lambda_R &= e^{i\vec{\xi}\cdot\left(i\frac{\vec{\sigma}}{2}\right) - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}} = e^{-i\frac{\vec{\sigma}}{2}\cdot(\vec{\theta} - i\vec{\xi})} \end{aligned}\right\}$$

One can also show that

$$[N_+^i, N_+^j] = i\varepsilon^{ijk} N_+^k$$

$$[N_-^i, N_-^j] = i\varepsilon^{ijk} N_-^k$$

N_+^i forms Lie algebra of $SU(2)$

N_-^i — , —

$\Rightarrow$ we separated $SO(3,1)$ into $SU(2)\otimes SU(2)$.

$\Rightarrow$ treat N_+^i as a separate "spin" operator

$\Rightarrow$ eigenvalues of $\vec{N}_+^2$ are $n_+(n_++1)$

with $n_+ = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

$\Rightarrow$ eigenvalues of $\vec{N}_-^2$ are $n_-(n_-+1)$

with $n_- = 0, \frac{1}{2}, \ldots$

(don't need hermiticity for this)

$\Rightarrow$ can classify representations of Lorentz group by (n_+, n_-)

$\Rightarrow \vec{J} = \vec{N}_+ + \vec{N}_- \Rightarrow$ the spin of the resulting field is given by $n_+ + n_-$

Under parity ($\mathbb{P}$) : $\vec{J} \to \vec{J}$, $\vec{K} \to -\vec{K}$ $\Rightarrow$

$\Rightarrow$ $\vec{N}_+ \to \vec{N}_-$ & $\vec{N}_- \to \vec{N}_+$.

they are not independent.

Classification of Fields.

$(0,0)$ spin-$\emptyset$, scalar field $\phi(x)$

$(\frac{1}{2},0)$ } spin-$\frac{1}{2}$ left-handed spinor $\chi_L(x)$ (2 d.o.f.)

$(0,\frac{1}{2})$ } right-handed spinor $\chi_R(x)$ (2 d.o.f.)

$(\frac{1}{2},\frac{1}{2})$ spin-1, 4-vector field $A_\mu(x)$
$= (\frac{1}{2},0)\otimes(0,\frac{1}{2}) \Rightarrow 2\times 2 = 4$ d.o.f.

$(1,0)$ spin-1, anti-symmetric self-dual tensor of rank-2, $B_{\mu\nu}$

$B_{\mu\nu} = -B_{\nu\mu}$, $B_{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}B_{\rho\sigma}$
3 d.o.f.

$(0,1)$ spin-1, — " — anti-self-dual — " —

$B_{\mu\nu} = -\frac{i}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}B_{\rho\sigma}$
3 d.o.f.

$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ has given parity, Dirac spinor

$(1,0)\oplus(0,1)$ e.g. Maxwell field strength $F_{\mu\nu}$
(also Kalb-Ramond field in strings!)

$(\frac{1}{2},\frac{1}{2}) \sim A_\mu$ e.g. Maxwell E&M, $A^\mu = (\Phi, \vec{A})$.

$(\frac{1}{2},0)\otimes(\frac{1}{2},0) = (0,0)\oplus(1,0) \Rightarrow (1,0)$ has 3 d.o.f.

Classification of fields

$(0,0)$	spin-0	scalar field $\varphi(x)$	1 d.o.f
$(\frac{1}{2},0)$ $(0,\frac{1}{2})$	spin-1/2	left-handed spinor $\chi_L = \binom{\chi_1}{\chi_2}$ right-handed spinor $\chi_R = \binom{\chi_3}{\chi_4}$ Dirac spinor $\Psi : (\frac{1}{2},0) \oplus (0,\frac{1}{2})$	2 d.o.f 2 d.o.f
$(\frac{1}{2},\frac{1}{2})$	spin-1	vector field $A_\mu(x)$ $(\frac{1}{2},\frac{1}{2}) = (\frac{1}{2},0) \otimes (0,\frac{1}{2})$	4 d.o.f.
$(1,0)$	spin-1	$B_{\mu\nu} = -B_{\nu\mu}$, $B_{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}B^{\rho\sigma}$	3 d.o.f
$(0,1)$	spin-1	$B_{\mu\nu} = -B_{\nu\mu}$, $B_{\mu\nu} = -\frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}B^{\rho\sigma}$	3 d.o.f.
$(1,\frac{1}{2}) \oplus (\frac{1}{2},1)$	spin-3/2	ψ^μ Rarita-Schwinger field $\gamma^\mu \psi_\mu = 0$ constr.	6+6 d.o.f. = 12 d.o.f. = 16 − 4
$(1,1)$	spin-2	$g_{\mu\nu}$ ~ graviton field $g^\mu{}_\mu = 4$ dim	9 d.o.f. = 10 − 1

$$\left(\text{BTW}, \quad A \oplus B = A \otimes \mathbb{1} + \mathbb{1} \otimes B \right)$$

d.o.f = degrees of freedom = # of independent <u>complex</u> components!

φ ~ can be complex, $\chi_{L,R}$ ~ complex, A_μ can be complex (e.g. W-boson), $B_{\mu\nu}$ is complex, ψ^μ —, — ...

$(1,1)$ spin-2, symmetric rank-2 tensor

e.g. $g_{\mu\nu}$ ~ gravitons, $T_{\mu\nu}$ ~ energy-momentum tensor

$3\times 3 = 9$ d.o.f. ($g_{\mu\nu}$ has 10 d.o.f, but as $g^{\mu}{}_{\mu} = 4 \Rightarrow$ really 9).

Spinor Representations

$\chi_L = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$, $\chi_R = \begin{pmatrix} \chi_3 \\ \chi_4 \end{pmatrix}$

Weyl spinors
$$\begin{cases} \chi_L(x) \to \chi'_L(x') \equiv \Lambda_L \, \chi_L(x) & \text{for } (\tfrac{1}{2}, 0) \\ \chi_R(x) \to \chi'_R(x') \equiv \Lambda_R \, \chi_R(x) & \text{for } (0, \tfrac{1}{2}) \end{cases}$$

$\chi_{L,R}$ have 2 components each $\Rightarrow \Lambda_{L,R}$ are 2×2 matrices

Generalizing $U = e^{i\vec{\xi}\cdot\vec{K} - i\vec{\theta}\cdot\vec{L}}$ from scalar field we write $\vec{L} \to \vec{J} = \vec{L} + \vec{S} = \vec{L} + \frac{1}{2}\vec{\sigma}$

where $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (su(2) property)

are Pauli matrices; also $\vec{K} = -i\left[x^0 \vec{\nabla} + \vec{x}\,\partial_0\right] - \frac{i}{2}\vec{\sigma}$.

$$\Rightarrow \chi'_L(x) = U_L(\Lambda)\,\chi_L(x) = e^{i\vec{\xi}\cdot\vec{K} - i\vec{\theta}\cdot\vec{J}}\,\chi_L(x)$$

$$= e^{i\vec{\xi}\cdot(-\frac{i}{2}\vec{\sigma}) - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}\,\chi_L(\Lambda^{-1}x)$$

$$\Rightarrow \chi'_L(\underbrace{x'}_{=\Lambda x}) = e^{-\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta} + i\vec{\xi})}\,\chi_L(x) = \Lambda_L\,\chi_L(x)$$