Last time

$$\boxed{\begin{aligned}\vec{L} &= -i\,\vec{x}\times\vec{\nabla} \\ \vec{K} &= -i\left[x^0\,\vec{\nabla} + \vec{x}\,\partial_0\right]\end{aligned}}$$

rotation generators

boost —"—

Such that

$$\boxed{e^{\frac{i}{2}\omega^{\mu\nu}L_{\mu\nu}} = e^{-i\vec{\xi}\cdot\vec{K} + i\vec{\theta}\cdot\vec{L}} = \Lambda}$$

← Lorentz transform.

Scalar field: $\phi(x) \to \phi'(x') = \phi(x)$, $x' = \Lambda x$

such that $\phi'(x) = \phi(\Lambda^{-1}x)$

Def. $U(\Lambda)$: $\boxed{\phi'(x) \equiv U(\Lambda)\,\phi(x)}$

Showed that $\boxed{U(\Lambda) = e^{-\frac{i}{2}\omega^{\mu\nu}L_{\mu\nu}} = e^{i\vec{\xi}\cdot\vec{K} - i\vec{\theta}\cdot\vec{L}}}$

What about particles with spin? In the spirit of angular momentum $\vec{J} = \vec{L} + \vec{S}$ we <u>defined</u>

$$\boxed{J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}}$$

where $[L_{\alpha\beta}, S_{\mu\nu}] = 0$ and $[S_{\mu\nu}, S_{\rho\sigma}]$ is defined by the same algebra as $L_{\alpha\beta}$.

Defining $J_i = \frac{1}{2}\varepsilon_{ijk}J_{jk}$ and $K_i = J_{0i}$ we got

$$\boxed{\begin{aligned}&[J_i, J_j] = i\,\varepsilon_{ijk}J_k \qquad [K_i, K_j] = -i\,\varepsilon_{ijk}J_k \\ &[J_i, K_j] = i\,\varepsilon_{ijk}K_k\end{aligned}}$$

Lorentz group algebra.

Def. $\vec{N}_\pm = \frac{1}{2}(\vec{J} \pm i\vec{K})$ (note: $\vec{J}, \vec{K}$ ~ hermitean, $\vec{N}_\pm$ are not)

We got

$$[N_{+i}, N_{+j}] = i\,\varepsilon_{ijk} N_{+k} \qquad su(2)$$

$$[N_{-i}, N_{-j}] = i\,\varepsilon_{ijk} N_{-k} \qquad su(2)$$

$$[N_{+i}, N_{-j}] = 0$$

We split Lorentz group into $SU(2) \otimes SU(2)$.

Analogy:

	Our $su(2)$	Ang. mom. operator in QM
operators	$\vec{N}_+$ (or $\vec{N}_-$)	$\vec{J}$
	$\vec{N}_+^{\,2}$	$\vec{J}^{\,2}$
eigenvalues	$n_+(n_+ + 1)$	$j(j+1)$
space it acts on	$\vert n_+\rangle$ representations of L. group of diff. spin	$\vert j\rangle$ ~ definite spin states

Classify all representations of $SO(3,1) = SU(2) \otimes SU(2)$

by eigenvalues of $\vec{N}_+^{\,2}$ & $\vec{N}_-^{\,2}$: (n_+, n_-).

Net spin: $s = n_+ + n_-$.

Classification of fields

$(0,0)$	spin-0	scalar field $\varphi(x)$	1 d.o.f
$(\frac{1}{2},0)$ $(0,\frac{1}{2})$	spin-1/2	left-handed spinor $\chi_L = \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}$ right-handed spinor $\chi_R = \begin{pmatrix}\chi_3\\ \chi_4\end{pmatrix}$ Dirac spinor Ψ : $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$	2 d.o.f 2 d.o.f
$(\frac{1}{2},\frac{1}{2})$	spin-1	vector field $A_\mu(x)$ $(\frac{1}{2},\frac{1}{2}) = (\frac{1}{2},0)\otimes(0,\frac{1}{2})$	4 d.o.f.
$(1,0)$	spin-1	$B_{\mu\nu} = -B_{\nu\mu}$, $B_{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}B^{\rho\sigma}$	3 d.o.f
$(0,1)$	spin-1	$B_{\mu\nu} = -B_{\nu\mu}$, $B_{\mu\nu} = -\frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}B^{\rho\sigma}$	3 d.o.f.
$(1,\frac{1}{2})\oplus(\frac{1}{2},1)$	spin-3/2	ψ^μ Rarita-Schwinger field $\gamma^\mu\psi_\mu = 0$ constr.	6+6 d.o.f = = 12 d.o.f. =16-4
$(1,1)$	spin-2	$g_{\mu\nu}$ ~ graviton field $\delta^\mu_\mu = 5$	9 d.o.f. =10-1

$$\left(\text{BTW,} \quad A \oplus B = A \otimes \mathbb{1} + \mathbb{1} \otimes B\right)$$

$(1,1)$ spin-2, symmetric rank-2 tensor

e.g. $g_{\mu\nu}$ ~ gravitons, $T_{\mu\nu}$ ~ energy-momentum tensor

$3\times 3 = 9$ d.o.f. ($g_{\mu\nu}$ has 10 d.o.f, but as $g^{\mu}{}_{\mu} = 4 \Rightarrow$ really 9).

Spinor Representations

$\chi_L = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$, $\chi_R = \begin{pmatrix} \chi_3 \\ \chi_4 \end{pmatrix}$

Weyl spinors
$$\begin{cases} \chi_L(x) \to \chi'_L(x') \equiv \Lambda_L \, \chi_L(x) & \text{for } (\tfrac{1}{2}, 0) \\ \chi_R(x) \to \chi'_R(x') \equiv \Lambda_R \, \chi_R(x) & \text{for } (0, \tfrac{1}{2}) \end{cases}$$

$\chi_{L,R}$ have 2 components each $\Rightarrow \Lambda_{L,R}$ are 2×2 matrices

Generalizing $U = e^{i\vec{\xi}\cdot\vec{K} - i\vec{\theta}\cdot\vec{L}}$ from scalar field we write $\vec{L} \to \vec{J} = \vec{L} + \vec{S} = \vec{L} + \frac{1}{2}\vec{\sigma}$

where $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (su(2) property)

are Pauli matrices; also $\vec{K} = -i\left[x^0 \vec{\nabla} + \vec{x}\,\partial_0\right] - \frac{i}{2}\vec{\sigma}$.

$$\Rightarrow \chi'_L(x) = U_L(\Lambda)\,\chi_L(x) = e^{i\vec{\xi}\cdot\vec{K} - i\vec{\theta}\cdot\vec{J}}\,\chi_L(x)$$

$$= e^{i\vec{\xi}\cdot(-\frac{i}{2}\vec{\sigma}) - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}\,\chi_L(\Lambda^{-1}x)$$

$$\Rightarrow \chi'_L(\underbrace{x'}_{=\Lambda x}) = e^{-\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta} + i\vec{\xi})}\,\chi_L(x) = \Lambda_L\,\chi_L(x)$$

$$\Rightarrow \boxed{\Lambda_L = e^{-\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}+i\vec{\xi})}}$$

satisfies $[N_+^i, N_+^j] = i\varepsilon^{ijk}N_+^k$

$$\vec{N}_+ = \frac{1}{2}(\vec{L}+i\vec{K}) + \frac{1}{2}\vec{\sigma} = \frac{1}{2}\underbrace{\vec{J}}_{=\vec{L}+\frac{\vec{\sigma}}{2}} + \frac{1}{2}\left(\frac{\vec{\sigma}}{2}+i\vec{K}\right) =$$

$$= \frac{1}{2}\vec{J} + \frac{i}{2}\underbrace{\left(\vec{K} - i\frac{\vec{\sigma}}{2}\right)}_{\text{new } \vec{K}} \quad \Rightarrow \text{had } -\frac{i}{2}\vec{\sigma} \text{ in } \vec{K}.$$

$\vec{N}_-$ ~ just $\vec{\xi} \xrightarrow{P} -\vec{\xi}$ $\Rightarrow$ $\boxed{\Lambda_R = e^{-\frac{i}{2}\vec{\sigma}(\vec{\theta}-i\vec{\xi})}}$ (note: $\vec{\sigma}$ in Λ_L & Λ_R operate in different spaces)

$\Rightarrow +\frac{i}{2}\vec{\sigma}$ in $\vec{K}$

$$\left(\vec{N}_- = \frac{1}{2}(\vec{L}-i\vec{K}) + \frac{1}{2}\vec{\sigma} = \frac{1}{2}\vec{J} + \frac{1}{2}\left(\frac{\vec{\sigma}}{2} - i\vec{K}\right) =\right.$$

$$\left. = \frac{1}{2}\vec{J} - \frac{i}{2}\underbrace{\left(\vec{K} + i\frac{\vec{\sigma}}{2}\right)}_{\text{new } \vec{K} \text{ for } \vec{N}_-}\right) \Rightarrow \text{gives } \Lambda_R \text{ above.}$$

Λ_L & Λ_R are <u>not</u> unitary:

$$\boxed{\Lambda_R^{+} = \Lambda_L^{-1}.} \quad \text{also} \quad \boxed{\Lambda_L^{+} = \Lambda_R^{-1}}$$

Def.

Require that the fields are parity eigenstates:

$(\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \Rightarrow$ <u>Dirac spinors</u>

$$\Psi_D = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$ 4-component object.

Under Lorentz transformation ψ_D becomes:

$$\boxed{\psi_D(x) \Rightarrow \psi_D'(x) = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} \psi_D(\Lambda^{-1}x).}$$

$$\begin{pmatrix} e^{-\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}+i\vec{\xi})} & 0 \\ 0 & e^{-\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}-i\vec{\xi})} \end{pmatrix}$$

Can we construct a Lorentz-invariant object?

$\psi^+\psi = \chi_L^+\chi_L + \chi_R^+\chi_R$ is not $\mathcal{L}$. inv., as

$\chi_L \to \Lambda_L\chi_L$, $\chi_L^+ \Rightarrow \chi_L^+\Lambda_L^+ = \chi_L^+\Lambda_R^{-1}$

$\Rightarrow \chi_L^+\chi_L = \chi_L^+\Lambda_R^{-1}\Lambda_L\chi_L$ not unitary $\Rightarrow$ not inv.

Def. Dirac γ-matrices (in Weyl representation):

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}}$$

$\boxed{\{A,B\} = AB + BA}$ anti-commutator.

$\Rightarrow$ Def. $\boxed{\bar{\psi} \equiv \psi^+\gamma^0}$ $\quad \bar{\psi}_\alpha = (\psi^+)_\beta(\gamma^0)_{\beta\alpha}$

$$\bar{\psi} = (\chi_L^+ \; \chi_R^+)\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = (\chi_R^+ \; \chi_L^+).$$

$$\bar{\Psi}\psi = \psi^+\gamma^0\psi = \begin{pmatrix}\chi_L^+ & \chi_R^+\end{pmatrix}\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\begin{pmatrix}\chi_L\\ \chi_R\end{pmatrix}$$

$$= \chi_L^+\chi_R + \chi_R^+\chi_L \quad \Rightarrow \quad \mathcal{L}.\text{ invariant!}$$

(check: $\chi_L^+\chi_R \to \chi_L^+ \underbrace{\Lambda_L^+}_{\Lambda_R^{-1}} \Lambda_R \chi_R = \chi_L^+\chi_R$!)

Def. $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix}-\mathbb{1} & 0\\ 0 & \mathbb{1}\end{pmatrix}$ (Weyl representation)

$$\Rightarrow \bar{\Psi}\gamma^5\psi = \begin{pmatrix}\chi_L^+ & \chi_R^+\end{pmatrix}\underbrace{\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\begin{pmatrix}-\mathbb{1} & 0\\ 0 & \mathbb{1}\end{pmatrix}}_{\begin{pmatrix}0 & +1\\ -1 & 0\end{pmatrix}}\begin{pmatrix}\chi_L\\ \chi_R\end{pmatrix}$$

$$= \chi_L^+\chi_R - \chi_R^+\chi_L \quad \Rightarrow \mathcal{L}.\text{ inv. too!}$$

but: $\mathbb{P}: \chi_L \to \chi_R, \quad \chi_R \to \chi_L \Rightarrow$

$\Rightarrow \mathbb{P}: \bar{\Psi}\gamma^5\psi \to -\bar{\Psi}\gamma^5\psi$ changes sign

$\Rightarrow \bar{\Psi}\gamma^5\psi \sim$ pseudoscalar.

$\Rightarrow$ $\bar{\Psi}\psi \sim$ Lorentz scalar

$\bar{\Psi}\gamma^5\psi \sim$ pseudoscalar

But: we need to find a Lagrangian $\Rightarrow$ need ∂_μ's

$\Rightarrow$ need vectors!