Last time: Classified all fields according to Lorentz group representations.

Spinor Representations (cont'd)

$$\chi_{\substack{L\\R}}(x) \to \chi'_{\substack{L\\R}}(x') \equiv \Lambda_{\substack{L\\R}}\, \chi_{\substack{L\\R}}(x)$$

$L = (\tfrac{1}{2}, 0)$

$R = (0, \tfrac{1}{2})$.

definition of $\Lambda_{L,R}$

Found that $\boxed{\Lambda_L = e^{-\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta} + i\vec{\xi})}}$ $\boxed{\Lambda_R = e^{-\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta} - i\vec{\xi})}}$

Note that $\Lambda_R^{\dagger} = \Lambda_L^{-1}$ and $\Lambda_L^{\dagger} = \Lambda_R^{-1}$: they are not hermitean!

Def. Dirac spinors $\psi_D = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$.

$$\psi_D(x) \to \psi_D'(x') = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} \psi_D(x).$$

Def. Dirac γ-matrices (Weyl repr.) : $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$

Note that defining anti-commutator by

Def. $\{A, B\} \equiv A B + B A$

one can show that $\boxed{\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}}$

Def. $\bar{\psi} = \psi^+ \gamma^0 = (\chi_L^+ \ \chi_R^+) \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = (\chi_R^+ \ \chi_L^+)$.

Showed that $\bar{\psi}\psi = \chi_L^+ \chi_R + \chi_R^+ \chi_L$ is Lorentz-invariant.

Under Lorentz transformation ψ_D becomes:

$$\psi_D(x) \Rightarrow \psi_D'(x) = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} \psi_D(\Lambda^{-1}x).$$

$$\begin{pmatrix} e^{-\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}+i\vec{\xi})} & 0 \\ 0 & e^{-\frac{i}{2}\vec{\sigma}\cdot(\vec{\theta}-i\vec{\xi})} \end{pmatrix}$$

Can we construct a Lorentz-invariant object?

$\psi^+\psi = \chi_L^+\chi_L + \chi_R^+\chi_R$ is not $\mathcal{L}$. inv., as

$\chi_L \to \Lambda_L\chi_L$, $\chi_L^+ \Rightarrow \chi_L^+\Lambda_L^+ = \chi_L^+\Lambda_R^{-1}$

$\Rightarrow \chi_L^+\chi_L = \chi_L^+\Lambda_R^{-1}\Lambda_L\chi_L$ not unitary $\Rightarrow$ not inv.

Def. Dirac γ-matrices (in Weyl representation):

$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$, $\boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}}$

$\boxed{\{A,B\} = AB + BA}$ anti-commutator.

$\Rightarrow$ **Def.** $\boxed{\bar{\psi} \equiv \psi^+\gamma^0}$ $\quad \bar{\psi}_\alpha = (\psi^+)_\beta(\gamma^0)_{\beta\alpha}$

$\bar{\psi} = (\chi_L^+\ \chi_R^+)\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = (\chi_R^+\ \chi_L^+)$.

$$\bar{\Psi}\psi = \psi^\dagger \gamma^0 \psi = \begin{pmatrix} \chi_L^\dagger & \chi_R^\dagger \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$$

$$= \chi_L^\dagger \chi_R + \chi_R^\dagger \chi_L \quad \Rightarrow \quad \mathcal{L}\text{. invariant!}$$

(check: $\chi_L^\dagger \chi_R \to \chi_L^\dagger \underbrace{\Lambda_L^\dagger}_{\Lambda_R^{-1}} \Lambda_R \chi_R = \chi_L^\dagger \chi_R$!)

Def. $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$ (Weyl representation)

$$\Rightarrow \bar{\Psi}\gamma^5\psi = \begin{pmatrix} \chi_L^\dagger & \chi_R^\dagger \end{pmatrix} \underbrace{\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}}_{\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$$

$$= \chi_L^\dagger \chi_R - \chi_R^\dagger \chi_L \quad \Rightarrow \mathcal{L}\text{. inv. too!}$$

but: $\mathbb{P}: \chi_L \to \chi_R, \quad \chi_R \to \chi_L \Rightarrow$

$\Rightarrow \mathbb{P}: \bar{\Psi}\gamma^5\psi \to -\bar{\Psi}\gamma^5\psi$ changes sign

$\Rightarrow \bar{\Psi}\gamma^5\psi \sim$ pseudoscalar.

$$\Rightarrow \boxed{\begin{array}{l} \bar{\Psi}\psi \sim \text{Lorentz scalar} \\ \bar{\Psi}\gamma^5\psi \sim \text{pseudoscalar} \end{array}}$$

But: we need to find a Lagrangian $\Rightarrow$ need ∂_μ's

$\Rightarrow$ need vectors!

What is a 4-vector field? How does it transform under Lorentz transformations?

If $\varphi(x)$ a scalar field $\Rightarrow$ $\partial_\mu \varphi(x) = A_\mu(x)$ is a 4-vector field.

$$\varphi(x) \to \varphi'(x') = \varphi(x)$$

$$\partial_\mu \varphi(x) \to \partial_\mu' \varphi'(x') = \partial_\mu' \varphi(x) = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} \varphi(x).$$

Now, $x'^\mu = \Lambda^\mu{}_\nu x^\nu \Rightarrow x' = \Lambda \cdot x \Rightarrow x = \Lambda^{-1} \cdot x'$

$$\Rightarrow x^\nu = (\Lambda^{-1})^\nu{}_\mu x'^\mu \Rightarrow \frac{\partial x^\nu}{\partial x'^\mu} = (\Lambda^{-1})^\nu{}_\mu .$$

As $\eta^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta^{\alpha\beta} \Rightarrow \delta^\mu{}_\nu = \Lambda^\mu{}_\alpha \Lambda_{\nu\beta} \eta^{\alpha\beta}$

$$= \Lambda^\mu{}_\alpha \Lambda_\nu{}^\alpha = \Lambda^\mu{}_\alpha \cdot (\Lambda^{-1})^\alpha{}_\nu \Rightarrow$$

$$(\Lambda^{-1})^\alpha{}_\nu = \Lambda_\nu{}^\alpha .$$

Thus $\frac{\partial x^\nu}{\partial x'^\mu} = (\Lambda^{-1})^\nu{}_\mu = \Lambda_\mu{}^\nu$

$$\Rightarrow \partial_\mu' \varphi'(x') = A'_\mu = \Lambda_\mu{}^\nu A_\nu \Rightarrow$$

$$\boxed{\begin{aligned} A_\mu \to A'_\mu &= \Lambda_\mu{}^\nu A_\nu \\ A^\mu \to A'^\mu &= \Lambda^\mu{}_\nu A^\nu \end{aligned}}$$ as expected!

Combine Dirac matrices into $\gamma^\mu = (\gamma^0, \vec{\gamma})$

$\mu = 0, 1, 2, 3.$

$\Rightarrow$ consider $\bar{\psi}\gamma^\mu\psi$. Claim: it's a 4-vector!

Check: rotations $\quad \psi \to \psi' = \begin{pmatrix} e^{-\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}} & 0 \\ 0 & e^{-\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}} \end{pmatrix} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$

$\Rightarrow \bar{\psi}\gamma^\mu\psi \Rightarrow$ 0th component is

$\bar{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi \Rightarrow$ invariant under rotations.

spatial component:

$$\bar{\psi}\gamma^i\psi = \psi^\dagger\gamma^0\gamma^i\psi = \psi^\dagger \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \psi =$$

$$= \psi^\dagger \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \psi = (\chi_L^\dagger \;\; \chi_R^\dagger) \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} =$$

$$= -\chi_L^\dagger\sigma^i\chi_L + \chi_R^\dagger\sigma^i\chi_R \longrightarrow -\chi_L^\dagger e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\sigma^i e^{-\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\chi_L$$

$$+ \chi_R^\dagger e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\sigma^i e^{-\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\chi_R \simeq (\text{infinitesimal}) =$$

$$= -\chi_L^\dagger \underbrace{\left(1 + \frac{i}{2}\vec{\sigma}\cdot\vec{\theta}\right)\sigma^i\left(1 - \frac{i}{2}\vec{\sigma}\cdot\vec{\theta}\right)} \chi_L + (L \to R)$$

$$= \sigma^i + \frac{i}{2}\theta^j \underbrace{[\sigma^j, \sigma^i]}_{2i\varepsilon^{jik}\sigma^k} + \ldots = \sigma^i + \varepsilon^{ijk}\theta^j\sigma^k$$

$\Rightarrow \bar{\psi}\gamma^i\psi \to \bar{\psi}\gamma^i\psi + \varepsilon^{ijk}\theta^j\bar{\psi}\gamma^k\psi$ ~ just like a rotation!

(clockwise, by angle θ)

One can show that the object $\bar{\psi}\gamma^\mu\psi$ transforms under boosts as expected of a 4-vector too $\Rightarrow$

$\bar{\psi}\gamma^\mu\psi$ is a 4-vector!

$\mathbb{P}$: $\bar{\psi}\gamma^0\psi = \psi^\dagger\psi = \chi_L^\dagger\chi_L + \chi_R^\dagger\chi_R$ ~ invariant

$$\bar{\psi}\gamma^i\psi = -\chi_L^\dagger\sigma^i\chi_L + \chi_R^\dagger\sigma^i\chi_R$$

$\Rightarrow$ $\bar{\psi}\gamma^i\psi \xrightarrow{\mathbb{P}} -\bar{\psi}\gamma^i\psi$ $\Rightarrow$ polar vector!

$\bar{\psi}\gamma^\mu\gamma^5\psi$ is a pseudo-vector (axial vector)

In general can show that

$$\psi \to \psi'_{(x')} = e^{-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}\psi(x)$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ ~ a reducible representation of Lorentz algebra

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Lagrangian for Dirac spinors:

$$\mathcal{L} = A\,\bar{\psi}\gamma^\mu\partial_\mu\psi + B\,\bar{\psi}\psi$$

L. invariants.

EOM: $\dfrac{\delta\mathcal{L}}{\delta\bar{\psi}} - \partial_\mu\underbrace{\left[\dfrac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{\psi})}\right]}_{=0} = 0 \Rightarrow (A\gamma^\mu\partial_\mu + B)\psi = 0$

$\Rightarrow$ act with $\gamma^\nu \partial_\nu$

$$\Rightarrow \left[A \underbrace{\gamma^\nu \gamma^\mu} \partial_\nu \partial_\mu + B \gamma^\nu \partial_\nu\right] \psi = 0$$

$$\tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}$$

$$\left[A \, \partial^2 + B \gamma^\nu \partial_\nu\right] \psi = 0$$

Now, $\gamma^\mu \partial_\mu \psi = -\frac{B}{A}\psi \Rightarrow \left[A \partial^2 - \frac{B^2}{A}\right]\psi = 0$

$$\Rightarrow \left[\partial^2 - \frac{B^2}{A^2}\right]\psi = 0$$

c.f. Klein-Gordon eqn: $\left[\partial^2 + m^2\right]\varphi = 0 \Rightarrow$ gives

$p^2 = m^2$ ~ correct on-shell condition

$\Rightarrow \frac{B^2}{A^2} = -m^2 \Rightarrow$ pick $A = i, \; B = -m$

$$\Rightarrow \boxed{\mathcal{L} = i \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi}\psi}$$

free Dirac field Lagrangian

EOM: $\boxed{\left[i \gamma^\mu \partial_\mu - m\right]\psi(x) = 0}$ Dirac equation.

Why no $\bar{\psi}\partial_\mu \partial^\mu \psi$ term in $\mathcal{L}$? For instance, dimensions:

$\dim \psi = 3/2 \Rightarrow \dim[\bar{\psi}\partial^2\psi] = M^5 \Rightarrow$ need dimensionful coupling.

$\Rightarrow$ not free field anymore.

(Ultimately experimental fact.)