Last time: Showed that vector fields transform as 4-vectors under Lorentz transformations:

$$\boxed{\begin{aligned} A_\mu &\to A'_{\mu}(x') = \Lambda_\mu{}^\nu A_\nu(x) \\ A^\mu &\to A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x) \end{aligned}}$$

We showed that $\bar{\psi}\psi$ is a scalar.

Def. $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\bar{\psi}\gamma^5\psi$ ~ pseudoscalar (changes sign under P).

We showed that $\bar{\psi}\gamma^\mu\psi$ is a 4-vector!

$\bar{\psi}\gamma^\mu\gamma^5\psi$ ~ pseudo-vector.

Dirac Lagrangian density

$$\boxed{\mathcal{L} = \bar{\psi}\left[i\gamma^\mu\partial_\mu - m\right]\psi}$$

EOM: $\dfrac{\delta\mathcal{L}}{\delta\bar{\psi}} = 0 \Rightarrow \boxed{\left[i\gamma^\mu\partial_\mu - m\right]\psi = 0}$ Dirac equation

Def. $\sigma_{\mu\nu} = \dfrac{i}{2}\left[\gamma_\mu, \gamma_\nu\right]$

$$\Rightarrow \psi(x) \to \psi'(x') = e^{-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}\psi(x)$$

under Lorentz transformations.

Dirac Lagrangian is

$$\mathcal{L} = \bar{\psi}\,[\,i\gamma^\mu \partial_\mu - m\,]\,\psi$$

Any symmetries? Yes, we have $\mathcal{L} \to \mathcal{L}$ under

$\psi \to e^{i\alpha}\psi$, $\quad \bar{\psi} \to e^{-i\alpha}\psi$, α a real number

$\Rightarrow \delta\mathcal{L} = 0 \Rightarrow$ remember we had for scalar fields

$$\delta\mathcal{L} = \sum_a \partial_\mu \left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu \varphi^a)} \delta\varphi^a \right]$$

$\Rightarrow$ similarly for spinors ψ & $\bar{\psi}$ we have

$$\delta\mathcal{L} = \partial_\mu \left[\underbrace{\frac{\delta\mathcal{L}}{\delta(\partial_\mu \psi)}}_{= \, i\bar{\psi}\gamma^\mu} \delta\psi + \underbrace{\frac{\delta\mathcal{L}}{\delta(\partial_\mu \bar{\psi})}}_{= \, 0} \delta\bar{\psi} \right] = 0$$

(as $\mathcal{L}$ is invariant under $U(1)$)

as nothing in $\mathcal{L}$ depends on $\partial_\mu \bar{\psi}$

$\Rightarrow$ get $\quad \partial_\mu \left[i\bar{\psi}\gamma^\mu \delta\psi \right] = 0$

$\delta\psi = (1 + i\alpha + \dots)\psi - \psi = i\alpha\psi \quad \Rightarrow$ get

$$\partial_\mu [\bar{\psi}\gamma^\mu\psi] = 0 \quad \Rightarrow \quad \boxed{j^\mu = \bar{\psi}\gamma^\mu\psi}$$ EM current.

is a conserved current : $\partial_\mu j^\mu = 0$. (can check explicitly)

In general can construct any bilinear object $\bar{\psi}\,\Gamma\,\psi$, with Γ a 4×4 matrix. "Full basis" of 4×4 matrices with definite Lorentz-transform properties is

$$\boxed{\Gamma = \{ \mathbb{1}, \gamma^\mu, \gamma^5, \gamma^\mu\gamma^5, \sigma^{\mu\nu} \}}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. 16 matrices.

One has:

bilinear	transformation law
$\bar{\psi}\psi$	scalar
$\bar{\psi}\gamma^5\psi$	pseudoscalar
$\bar{\psi}\gamma^\mu\psi$	vector
$\bar{\psi}\gamma^\mu\gamma^5\psi$	axial vector
$\bar{\psi}\,\sigma^{\mu\nu}\,\psi$	antisymmetric tensor

$j_\mu^5 = \bar{\psi}\gamma^5\gamma_\mu\psi$ is also a 4-vector $\begin{pmatrix}\text{axial}\\ \text{current}\end{pmatrix}$

Is it conserved? In fact $\partial_\mu j^{\mu 5} = 2im\bar{\psi}\gamma^5\psi$

$\Rightarrow$ it is conserved only if $m=0$.

<u>Energy-momentum tensor</u>: $T_{\mu\nu} = \dfrac{\delta\mathcal{L}}{\delta(\partial^\mu\psi)}\partial_\nu\psi + \cancel{\dfrac{\delta\mathcal{L}}{\delta(\partial^\mu\bar{\psi})}}^{\,0}\partial_\nu\bar{\psi} -$

$- g_{\mu\nu}\mathcal{L}$ by analogy with scalar field.

We get $T_{\mu\nu} = i\bar{\Psi}\gamma_\mu \partial_\nu \Psi - g_{\mu\nu}\left[\bar{\Psi}(i\gamma^\alpha\partial_\alpha - m)\Psi\right]$

$$\Rightarrow \quad T_{\mu\nu} = \bar{\Psi}\left[i\gamma_\mu\partial_\nu - g_{\mu\nu}\, i\gamma^\alpha\partial_\alpha + g_{\mu\nu}\, m\right]\Psi$$

However, we can simplify this by using Dirac equation $(i\gamma^\alpha\partial_\alpha - m)\Psi = 0$ $\Rightarrow$ get

$$T_{\mu\nu} = i\bar{\Psi}\gamma_\mu\partial_\nu\Psi \qquad \text{(not symmetric)}$$

Remember that the Hamiltonian $H = \int d^3x\, T_{00}$.

We get $H = \int d^3x\; i\underbrace{\bar{\Psi}}_{\Psi^\dagger \underbrace{\gamma^0\gamma^0}_{\mathbb{1}}}\gamma^0\partial_t\Psi = \int d^3x\; i\Psi^\dagger\partial_t\Psi$

$$\Rightarrow \quad H = \int d^3x\; i\Psi^\dagger\partial_t\Psi \qquad \text{problem: } H \text{ is not} \geq 0!$$

(This is different from scalar fields, for which H was ≥ 0 for the field!)

$T_{\mu\nu}$ can be symmetrized: $T^{\text{symm.}}_{\mu\nu} = i\bar{\Psi}\left[\frac{1}{2}\left(\gamma_\mu\overleftrightarrow{\partial}_\nu + \gamma_\nu\overleftrightarrow{\partial}_\mu\right)\right]\Psi$

$\sim$ can show that $\partial^\mu T^{\text{symm}}_{\mu\nu} = 0$.

Here $\bar{\xi}\overleftrightarrow{\partial}_\mu\Psi = \bar{\xi}\,\partial_\mu\Psi - (\partial_\mu\bar{\xi})\Psi$.

Useful γ-matrix formulas:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \qquad (\gamma^0)^\dagger = \gamma^0, \ (\gamma^i)^\dagger = -\gamma^i$$

" $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \quad , \quad (\gamma^5)^\dagger = \gamma^5$$

$$(\gamma^0)^2 = -(\gamma^i)^2 = \mathbb{1}, \quad (\gamma^5)^2 = \mathbb{1}.$$

$$\{\gamma^5, \gamma^\mu\} = 0$$

easy to see that $(\gamma^\mu)^2 = g^{\mu\mu}$ (no summation)

(Easy to check.)

Also $\gamma_\mu\gamma^\mu = 4$, $\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$.

Finally, note that $\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$ in Weyl basis

$\Rightarrow$ Def. $\boxed{P_L = \frac{\mathbb{1} - \gamma^5}{2}, \quad P_R = \frac{\mathbb{1} + \gamma^5}{2}}$

$$\Rightarrow P_L \psi = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix} \equiv \psi_L$$

projection on left-handed spinor

$$P_R \psi = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}\begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \begin{pmatrix} 0 \\ \chi_R \end{pmatrix} \equiv \psi_R$$

-"- right-handed

Can check that $P_L^2 = P_L$, $P_R^2 = P_R$, $P_L P_R = P_R P_L = 0$.

($\psi_L \sim$ helicity $-1/2$, $\psi_R \sim$ helicity $+1/2$, more later)

Take Dirac equation $\left[i\gamma^\mu\partial_\mu - m\right]\psi(x) = 0$.

In momentum space $\psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x}\psi(p)$

$\Rightarrow (\gamma^\mu p_\mu - m)\psi(p) = 0$; If $m=0 \Rightarrow \gamma^\mu p_\mu \psi(p) = 0$

As $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ & $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$ $\Rightarrow$

$\Rightarrow \underset{\gamma^0 p_0 + \gamma^i p_i}{\overset{\shortparallel}{\gamma^\mu p_\mu}} = \begin{pmatrix} 0 & p_0 - \vec{p}\cdot\vec{\sigma} \\ p_0 + \vec{p}\cdot\vec{\sigma} & 0 \end{pmatrix} \Rightarrow$ Dirac equation becomes:

$$\begin{pmatrix} 0 & p_0 - \vec{p}\cdot\vec{\sigma} \\ p_0 + \vec{p}\cdot\vec{\sigma} & 0 \end{pmatrix}\begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = 0$$

Def.

Helicity operator $h \equiv \frac{\vec{p}\cdot\vec{S}}{|\vec{p}|}$ $\Rightarrow$ for spin-$\frac{1}{2}$ particles have $\vec{S} = \frac{1}{2}\vec{\sigma} \Rightarrow h = \frac{1}{2|\vec{p}|}\vec{p}\cdot\vec{\sigma}$.

Physical meaning: projection of spin on $\vec{p}$ direction.

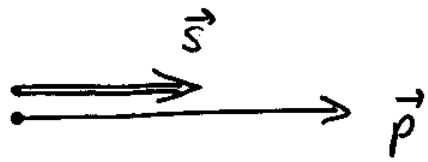
We get
$$(p_0 - \vec{p}\cdot\vec{\sigma})\chi_R = 0$$
$$(p_0 + \vec{p}\cdot\vec{\sigma})\chi_L = 0$$

Hence, as $|\vec{p}| = p^0$ we get
$$h\chi_R = +1/2$$
$$h\chi_L = -1/2$$

$\Rightarrow$ χ_R has helicity $+1$

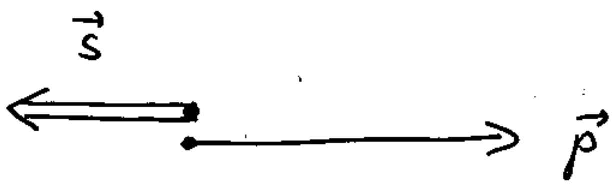
χ_L —//— -1

$\Rightarrow$ χ_R is called right-handed as



the spin is $\upuparrows$ to $\vec{p}$. (hence $h = +1/2$)

$\Rightarrow$ χ_L is called left-handed as



the spin is $\downarrow\uparrow$ to $\vec{p}$ (hence $h = -1/2$).

Poincare Group

add space-time translations: $x^\mu \to x'^\mu = x^\mu + a^\mu$

Def. Poincare group is a group of Lorentz transformations $\Lambda^\mu{}_\nu$ and translations a^μ:

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

$\Lambda^\mu{}_\nu$ = 6 parameters (boosts & rotations)

a^μ = 4 parameters

Total = 10 parameters.

<u>Generators</u>:
$$\begin{cases} J_{\mu\nu} = i[x_\mu \partial_\nu - x_\nu \partial_\mu] + S_{\mu\nu} \\ P_\mu = i\partial_\mu \end{cases}$$

$P_\mu = i\partial_\mu$ ~ generators of translations.

$P_0 = i\partial_0$; $\quad P^i = i\partial^i = -i\partial_i = -i\nabla_i$

$\Rightarrow \vec{P} = -i\vec{\nabla}$ ~ 3-momentum operator

<u>Poincare algebra</u>:

$$[P_\mu, P_\nu] = 0$$
$$[P_\mu, J_{\rho\sigma}] = i(g_{\mu\rho} P_\sigma - g_{\mu\sigma} P_\rho)$$
$[J_{\mu\nu}, J_{\rho\sigma}]$ = same as before

Operator $P_\mu P^\mu$ commutes with everything

$\Rightarrow$ Casimir operator.

Def. Pauli-Lubanski vector $\boxed{w^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} P_\nu J_{\rho\sigma}}$

$\Rightarrow$ can show that $w^\mu = -\frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu S_{\rho\sigma}$. (or, in rest frame of particle $w_i = -m S_i$, $w_0 = 0$)

$w_\mu w^\mu$ is another Casimir operator.

Representations are:

(1) $P_\mu P^\mu = m^2 > 0 \Rightarrow w_\mu w^\mu = -m^2 s(s+1)$ (can show; s: spin)

$s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

(2) $P_\mu P^\mu = 0 \Rightarrow W_\mu W^\mu = 0$, $W_\mu P^\mu = 0$ (always)

$\Rightarrow w^\mu = h P^\mu \Rightarrow h = 0, \pm\frac{1}{2}, \pm 1, \dots \sim$ helicity.

in general helicity $h = \pm s$, s is spin, discretized.

(e.g. photon has 2 polarizations corresponding to $h = \pm 1$, neutrino has $h = \pm 1$, ...)

(3) $P_\mu P^\mu = 0$ but s in continuous. Possible, but not realized in nature.