Last time: Canonical Quantization (cont'd)

Real Scalar Field (cont'd)

By analogy with Quantum Mechanics we've constructed a way to quantize fields:

$$[\varphi(\vec{x},t), \pi(\vec{y},t)] = i\,\delta^3(\vec{x}-\vec{y})$$
$$[\varphi(\vec{x},t), \varphi(\vec{y},t)] = [\pi(\vec{x},t), \pi(\vec{y},t)] = 0$$

$\pi = \dfrac{\delta \mathcal{L}}{\delta \dot{\varphi}}$ is the canonical momentum.

Time evolution is accomplished by Hamiltonian operator: $H = \int d^3x\,[\dot{\varphi}\,\pi - \mathcal{L}]$.

Heisenberg picture: operators are time-dependent

$$\left.\begin{aligned} \phi_H(\vec{x},t) &= e^{iHt}\,\phi_S(\vec{x},0)\,e^{-iHt} \\ \pi_H(\vec{x},t) &= e^{iHt}\,\pi_S(\vec{x},0)\,e^{-iHt} \end{aligned}\right\} \text{ as } -i\partial_0\phi = [H,\phi].$$

States are time-independent. $|\psi\rangle_H$.

Schroedinger picture: Operators are time-independent, $\phi_S(\vec{x})$, states are time-dependent: $|\psi,t\rangle_S = e^{-iHt}\,|\psi,t=0\rangle_S$
$= e^{-iHt}\,|\psi\rangle_H$

Interaction picture: $H = H_0 + H_{int} \Rightarrow$

$$\phi_I(\vec{x}, t) = e^{iH_0 t} \, \phi_S(\vec{x}) \, e^{-iH_0 t}$$

$$|\psi, t\rangle_I = e^{-iH_{int} t} \, |\psi, t=0\rangle_I = e^{-iH_{int} t} \, |\psi\rangle_H$$

$$= e^{-iH_{int} t} \, e^{iHt} \, |\psi, t\rangle_S$$

Hamiltonian for real scalar theory with $\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \, \partial^\mu \varphi - \frac{m^2}{2}\varphi^2$

is $$H = \int d^3x \left[\frac{1}{2} \pi^2 + \frac{1}{2} (\vec{\nabla} \varphi)^2 + \frac{m^2}{2} \varphi^2 \right].$$

Generates time evolution: $\partial_0 \pi = i [H, \pi]$ (Heisenberg picture)

$\pi = \dot{\varphi} \Rightarrow \partial_0^2 \varphi = i [H, \pi] = i \left[\int d^3y \left[\frac{\pi^2(y)}{2} + \frac{1}{2} (\vec{\nabla} \varphi(y))^2 + \right.\right.$

$\left.\left. + \frac{m^2}{2} \varphi^2(y) \right], \pi(x) \right]$, where $x^\mu = (t, \vec{x})$, $y^\mu = (t, \vec{y})$.

$[\pi^2, \pi] = 0$ as π's commute.

$[\varphi^2(y), \pi(x)]$: use $[A, BC] = [A, B]C + B[A, C]$

to write

$[\varphi^2(y), \pi(x)] = -[\pi(x), \varphi^2(y)] = -[\pi(x), \varphi(y)] \varphi(y) -$

$-\varphi(y) [\pi(x), \varphi(y)] = i\delta(\vec{x} - \vec{y}) \varphi(y) + i\delta(\vec{x} - \vec{y}) \varphi(y) =$

$= 2i \, \varphi(x) \, \delta(\vec{x} - \vec{y})$

The remaining commutator is a bit more subtle:

$$\left[\left(\vec{\nabla}\varphi(y)\right)^2, \pi(x) \right] = - \left[\pi(x), \left(\vec{\nabla}\varphi(y)\right)^2 \right] =$$

$$= - \left[\pi(x), \vec{\nabla}\varphi(y) \right] \cdot \vec{\nabla}\varphi(y) - \vec{\nabla}\varphi(y) \cdot \left[\pi(x), \vec{\nabla}\varphi(y) \right] =$$

$$= \vec{\nabla}_y \left(\left[\varphi(y), \pi(x) \right] \right) \cdot \vec{\nabla}\varphi(y) + \vec{\nabla}\varphi(y) \cdot \vec{\nabla}_y \left(\left[\varphi(y), \pi(x) \right] \right)$$

$$= 2i \left[\vec{\nabla}_y \delta(\vec{x} - \vec{y}) \right] \cdot \vec{\nabla}\varphi(y).$$

Hence we get

$$\partial_0^2 \varphi = i \int d^3y \left[\frac{1}{\cancel{2}} \cdot \cancel{2} i \left(\vec{\nabla}_y \delta(\vec{x} - \vec{y}) \right) \cdot \vec{\nabla}\varphi(y) + \frac{m^2}{\cancel{2}} \cdot \cancel{2} i \, \varphi(x) \delta(\vec{x} - \vec{y}) \right]$$

$$= + \int d^3y \, \delta(\vec{x} - \vec{y}) \, \vec{\nabla}^2 \varphi(y) - \varphi(x) m^2 =$$

$$= \vec{\nabla}^2 \varphi - m^2 \varphi$$

$$\Rightarrow \left[\partial_0^2 - \vec{\nabla}^2 + m^2 \right] \varphi = 0 \quad \Rightarrow$$

Klein-Gordon eq'n holds at the operator level!

Hence we wrote

$$\varphi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \left[\hat{a}_{\vec{k}} \, e^{-ik \cdot x} + \hat{a}^+_{\vec{k}} \, e^{ik \cdot x} \right]$$

and showed that

$$\left[\hat{a}_{\vec{k}}, \hat{a}^+_{\vec{k}'} \right] = (2\pi)^3 \, 2\varepsilon_k \, \delta^3(\vec{k} - \vec{k}')$$

$$\left[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'} \right] = \left[\hat{a}^+_{\vec{k}}, \hat{a}^+_{\vec{k}'} \right] = 0$$

For free scalar field we have

$$H = \int d^3x \left[\frac{1}{2} \pi^2 + \frac{1}{2} (\vec{\nabla} \varphi)^2 + \frac{m^2}{2} \varphi^2 \right]$$

Plug in φ, $\pi = \dot{\varphi}$ $\Rightarrow$

$$H = \int d^3x \cdot \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \frac{d^3k'}{(2\pi)^3 2\varepsilon_{k'}} \Big\{ \hat{a}_{\vec{k}} \hat{a}_{\vec{k}'} e^{-ik\cdot x - ik'\cdot x} \cdot$$

$$\cdot \left[-\frac{\varepsilon_k \varepsilon_{k'}}{2} - \frac{1}{2} \vec{k}\cdot\vec{k}' + \frac{m^2}{2} \right] + \hat{a}^+_{\vec{k}} \hat{a}^+_{\vec{k}'} e^{ik\cdot x + ik'\cdot x} \cdot$$

$$\cdot \left[-\frac{\varepsilon_k \varepsilon_{k'}}{2} - \frac{\vec{k}\cdot\vec{k}'}{2} + \frac{m^2}{2} \right] + \hat{a}_{\vec{k}} \hat{a}^+_{\vec{k}'} e^{-ik\cdot x + ik'\cdot x} \cdot$$

$$\cdot \left[\frac{\varepsilon_k \varepsilon_{k'}}{2} + \frac{\vec{k}\cdot\vec{k}'}{2} + \frac{m^2}{2} \right] + \hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}'} e^{ik\cdot x - ik'\cdot x} \cdot$$

$$\cdot \left[\frac{\varepsilon_k \varepsilon_{k'}}{2} + \frac{\vec{k}\cdot\vec{k}'}{2} + \frac{m^2}{2} \right] \Big\} =$$

integrate d^3x $\Rightarrow$ get $(2\pi)^3 \delta(\vec{k}+\vec{k}')$ for 1st 2 terms and $(2\pi)^3 \delta(\vec{k}-\vec{k}')$ for last 2 terms.

$$= \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \frac{d^3k'}{(2\pi)^3 2\varepsilon_{k'}} \Big\{ \Big[\hat{a}_{\vec{k}} \hat{a}_{-\vec{k}} e^{-i\varepsilon_k t - i\varepsilon_k t} + \hat{a}^+_{\vec{k}} \hat{a}^+_{-\vec{k}} \cdot$$

$$\cdot e^{2i\varepsilon_k t} \Big] \underbrace{\left[-\frac{1}{2} \varepsilon_k^2 + \frac{1}{2} (\vec{k}^2 + m^2) \right]}_{=0} (2\pi)^3 \delta(\vec{k}+\vec{k}') +$$

$$+ \left[\hat{a}_{\vec{k}} \hat{a}^+_{\vec{k}} + \hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}} \right] \left[\frac{1}{2} \varepsilon_k^2 + \frac{1}{2} \overbrace{(\vec{k}^2 + m^2)}^{\varepsilon_k^2} \right] (2\pi)^3 \delta(\vec{k}-\vec{k}') \Big\} =$$

= (integrating over $\vec{k}'$ is trivial) =

$$= \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k}\ \frac{1}{2\cancel{\varepsilon_k}} \cdot \varepsilon_k^{\cancel{2}} \cdot \left[\hat{a}_{\vec{k}}\, \hat{a}^{+}_{\vec{k}} + \hat{a}^{+}_{\vec{k}}\, \hat{a}_{\vec{k}} \right]$$

Finally, $$H = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k}\ \frac{\varepsilon_k}{2} \left[\hat{a}_{\vec{k}}\, \hat{a}^{+}_{\vec{k}} + \hat{a}^{+}_{\vec{k}}\, \hat{a}_{\vec{k}} \right]$$

aside:

Note that while φ, π were time-dependent (and hence in Heisenberg picture), $\hat{a}_{\vec{k}}$ & $\hat{a}^{+}_{\vec{k}}$ are not and are in Schroedinger picture.

One can show that $\hat{a}^{H}_{\vec{k}}(t) = e^{iHt}\, \hat{a}^{S}_{\vec{k}}\, e^{-iHt} =$

$$= e^{-i\varepsilon_k t}\ \hat{a}^{S}_{\vec{k}}$$

and $$\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k} \left[\hat{a}^{H}_{\vec{k}}(t)\, e^{i\vec{k}\cdot\vec{x}} + \hat{a}^{+}_{\vec{k}\,H}(t)\, e^{-i\vec{k}\cdot\vec{x}} \right].$$

Now, as $\left[\hat{a}_{\vec{k}}, \hat{a}^{+}_{\vec{k}'} \right] = (2\pi)^3\, 2\varepsilon_k\, \delta^3(\vec{k}-\vec{k}')$, we have

$$H = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k}\ \varepsilon_k \left[\hat{a}^{+}_{\vec{k}}\, \hat{a}_{\vec{k}} + \underbrace{\frac{1}{2} \cdot (2\pi)^3\, 2\varepsilon_k\, \delta^3(\vec{0})}_{\text{sick infinity!}} \right]$$

$$\Rightarrow H = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k}\ \varepsilon_k\ \hat{a}^{+}_{\vec{k}}\, \hat{a}_{\vec{k}} + \infty .$$

$\infty \sim$ just a constant (for $\forall \vec{k}$) $\Rightarrow$ drop (zero point energy).

only gravity would see this $\infty \Rightarrow$ don't talk about it here

Def. Particle number operator $\boxed{\hat{N}(\vec{k}) \equiv \hat{a}^{+}_{\vec{k}} \hat{a}_{\vec{k}}}$

Write $$\boxed{H = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k}\; \varepsilon_k \hat{N}(\vec{k})}$$

Hence H = Net Energy = energy of one particle $\otimes$ # particles.

Def.
Total # of particles $\hat{N} \equiv \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k} \hat{N}(\vec{k})$.

Classify all states by eigenvalues of $\hat{N}$:

$$\hat{N}\,|n\rangle = n\,|n\rangle$$

Now, $[\hat{N}, \hat{a}^{+}_{\vec{k}}] = \int \frac{d^3k'}{(2\pi)^3 2\varepsilon_{k'}} [\hat{a}^{+}_{\vec{k}'} \hat{a}_{\vec{k}'}, \hat{a}^{+}_{\vec{k}}] =$

$$= \int \frac{d^3k'}{(2\pi)^3 2\varepsilon_{k'}} \Big[\hat{a}^{+}_{\vec{k}'} \underbrace{\hat{a}_{\vec{k}'} \hat{a}^{+}_{\vec{k}}}_{= \hat{a}^{+}_{\vec{k}} \hat{a}_{\vec{k}'} + (2\pi)^3\, 2\varepsilon_k\, \delta^3(\vec{k}-\vec{k}')} - \hat{a}^{+}_{\vec{k}} \hat{a}^{+}_{\vec{k}'} \hat{a}_{\vec{k}'}\Big] = \hat{a}^{+}_{\vec{k}}\,.$$

Hence
$$\boxed{\begin{aligned} [\hat{N}, \hat{a}^{+}_{\vec{k}}] &= \hat{a}^{+}_{\vec{k}} \\ [\hat{N}, \hat{a}_{\vec{k}}] &= -\hat{a}_{\vec{k}} \end{aligned}}$$
← can be also shown.

$$\hat{N}\, \hat{a}^+_{\vec{k}} |n\rangle = \left(\hat{a}^+_{\vec{k}} \hat{N} + \hat{a}^+_{\vec{k}}\right) |n\rangle = (n+1)\, \hat{a}^+_{\vec{k}} |n\rangle$$

$\Rightarrow$ state $\hat{a}^+_{\vec{k}} |n\rangle$ has $(n+1)$-particles $\Rightarrow$

$\Rightarrow$ $\hat{a}^+_{\vec{k}}$ is a creation operator for a particle of momentum $\vec{k}$ & energy $\mathcal{E}_k$

$\hat{a}_{\vec{k}}$ is an annihilation operator —,,—

as $\hat{N}\, \hat{a}_{\vec{k}} |n\rangle = \left(\hat{a}_{\vec{k}} \hat{N} - \hat{a}_{\vec{k}}\right)|n\rangle = (n-1)\, \hat{a}_{\vec{k}} |n\rangle$

particles $\geq 0 \Rightarrow \langle n | \hat{N}_{\vec{k}} | n\rangle \geq 0$

$\left(\text{as } \langle n | \hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}} | n\rangle \geq 0\right) \Rightarrow \quad n(\vec{k}) \langle n | n\rangle \geq 0$

as $a_{\vec{k}}$ turns $n \to n-1 \Rightarrow$ there must be a ground state, otherwise would get $n<0$.

$$\hat{a}_{\vec{k}} |0\rangle = 0$$ ~ ground state (vacuum) (for any $\vec{k}$)

$\hat{N}_{\vec{k}} |0\rangle = 0 \Rightarrow \hat{N} |0\rangle = 0$. ~ zero particles in ground state

$$|\vec{k}\rangle = \hat{a}^+_{\vec{k}} |0\rangle$$ single-particle state

$$\langle \vec{k}' | \vec{k} \rangle = \langle 0 | \hat{a}_{\vec{k}'} \hat{a}^+_{\vec{k}} | 0\rangle = (2\pi)^3\, 2\mathcal{E}_k\, \delta^3(\vec{k} - \vec{k}')$$

normalization

$|\vec{k}_1, \vec{k}_2\rangle = \hat{a}^+_{\vec{k}_1} \hat{a}^+_{\vec{k}_2} |0\rangle$ two-particle state

$H|\vec{k}_1, \vec{k}_2\rangle = (\varepsilon_{k_1} + \varepsilon_{k_2}) |\vec{k}_1, \vec{k}_2\rangle$

In general $|\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_n\rangle = \hat{a}^+_{\vec{k}_1} \hat{a}^+_{\vec{k}_2} \cdots \hat{a}^+_{\vec{k}_n} |0\rangle$

n-particle state. (Fock states)

Any state of the theory can be expanded into Fock states:

$$\boxed{|\Psi\rangle = c_0|0\rangle + \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} c_{\vec{k}} |\vec{k}\rangle + \int \frac{d^3k_1\, d^3k_2}{(2\pi)^3 2\varepsilon_{k_1} (2\pi)^3 2\varepsilon_{k_2}} \cdot c_{\vec{k}_1, \vec{k}_2} |\vec{k}_1, \vec{k}_2\rangle + \ldots}$$

One-particle wave function:

$$\Psi(x) \equiv \langle 0| \varphi(x) |\vec{k}\rangle$$

$$\Psi(x) = \langle 0| \int \frac{d^3p}{(2\pi)^3 2\varepsilon_p} \Big[\hat{a}_{\vec{p}}\, e^{-ip\cdot x} + \underbrace{\cancel{\hat{a}^+_{\vec{p}}\, e^{ip\cdot x}}}_{0}\Big] \underbrace{|\vec{k}\rangle}_{\hat{a}^+_{\vec{k}}|0\rangle}$$

$$= \langle 0| \int \frac{d^3p}{(2\pi)^3 2\varepsilon_p} e^{-ip\cdot x} \underbrace{[\hat{a}_{\vec{p}}, \hat{a}^+_{\vec{k}}]}_{(2\pi)^3\, 2\varepsilon_p\, \delta^3(\vec{k}-\vec{p})} |0\rangle = e^{-ik\cdot x}$$

just a plane wave, as expected in free field theory.