Last time: we quantized complex scalar field:

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi$$

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \left[\hat{a}_{\vec{k}} e^{-ik\cdot x} + \hat{b}^+_{\vec{k}} e^{ik\cdot x} \right]$$

$$[\hat{a}_{\vec{k}}, \hat{a}^+_{\vec{k}'}] = (2\pi)^3 \, 2\varepsilon_k \, \delta(\vec{k}-\vec{k}')$$
$\quad b \quad b$

Hamiltonian $\quad H = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \varepsilon_k \left[\hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}} + \hat{b}^+_{\vec{k}} \hat{b}_{\vec{k}} \right]$

Conserved $U(1)$ current $\quad (\varphi \to e^{i\alpha}\varphi)$

$$j^\mu = i\left[\varphi \partial^\mu \varphi^* - \varphi^* \partial^\mu \varphi \right],$$

$Q = \int d^3x \, j^0(\vec{x},t)$ ~ conserved charge

Get $\quad Q = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \left[\underbrace{\hat{a}^+_{\vec{k}} \hat{a}_{\vec{k}}}_{\text{charge } +1} - \underbrace{\hat{b}^+_{\vec{k}} \hat{b}_{\vec{k}}}_{\text{charge } -1} \right]$

$\hat{a}^+_{\vec{k}}$ ~ creates particles

$\hat{b}^+_{\vec{k}}$ ~ creates anti-particles.

Quantization of Spinor Field (cont'd)

$\psi_{Dirac} = S\, \psi_{Weyl}$, $\quad SS^+ = S^+S = \mathbb{1}$

$\gamma^\mu_{Dirac} = S\, \gamma^\mu_{Weyl} S^{-1} \Rightarrow \mathcal{L}$ is invariant

$\Rightarrow$ choosing $S = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix}$ we got

$$\gamma^0_{\text{Dirac}} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^i_{\text{Dirac}} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \text{(Dirac basis)}$$

$$\gamma^5_{\text{Dirac}} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

$$\psi_{\text{Dirac}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_R + \chi_L \\ \chi_R - \chi_L \end{pmatrix}.$$

Solution of Dirac Equation (cont'd).

$(i\gamma^\mu \partial_\mu - m)\psi = 0 \Rightarrow$ can show that $[\Box + m^2]\psi = 0$

$$\Rightarrow \psi(x) = \int \frac{d^3k}{(2\pi)^3 2E_k} \left[e^{-ik\cdot x} \psi^{(+)}(\vec{k}) + e^{ik\cdot x} \psi^{(-)}(\vec{k}) \right].$$

$\Rightarrow$ plug in to get $\begin{cases} (\gamma\cdot k - m)\psi^{(+)}(\vec{k}) = 0 \\ (\gamma\cdot k + m)\psi^{(-)}(\vec{k}) = 0 \end{cases}$

Solved to write

$$\psi^{(+)} = \begin{pmatrix} \psi_u^{(+)} \\ \frac{\vec{\sigma}\cdot\vec{k}}{E_k + m} \psi_u^{(+)} \end{pmatrix}, \qquad \psi^{(-)} = \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{E_k + m} \psi_\ell^{(-)} \\ \psi_\ell^{(-)} \end{pmatrix}$$

with $\psi_u^{(+)}$, $\psi_\ell^{(-)}$ a arbitrary 2-component objects

$$\Rightarrow \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \Rightarrow$$

$$\gamma \cdot k = \gamma^0 k_0 + \gamma^i k_i = \gamma_0 k_0 - \vec{\gamma} \cdot \vec{k} = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}$$

$$- \begin{pmatrix} 0 & \vec{\sigma} \cdot \vec{k} \\ -\vec{\sigma} \cdot \vec{k} & 0 \end{pmatrix} = \begin{pmatrix} \varepsilon & -\vec{\sigma} \cdot \vec{k} \\ \vec{\sigma} \cdot \vec{k} & -\varepsilon \end{pmatrix}$$

$$\Rightarrow (\gamma \cdot k - m)\, \psi^{(+)} = \begin{pmatrix} \varepsilon - m & -\vec{\sigma} \cdot \vec{k} \\ \vec{\sigma} \cdot \vec{k} & -\varepsilon - m \end{pmatrix} \begin{pmatrix} \psi_u^{(+)} \\ \psi_\ell^{(+)} \end{pmatrix} = 0$$

$$\begin{cases} (\varepsilon - m)\, \psi_u^{(+)} - \vec{\sigma} \cdot \vec{k}\, \psi_\ell^{(+)} = 0 \\ \vec{\sigma} \cdot \vec{k}\, \psi_u^{(+)} - (\varepsilon + m)\, \psi_\ell^{(+)} = 0 \end{cases} \Rightarrow \psi_\ell^{(+)} = \frac{\vec{\sigma} \cdot \vec{k}}{\varepsilon + m} \psi_u^{(+)}$$

solves the whole thing (why?)

$$\Rightarrow \psi^{(+)} = \begin{pmatrix} \psi_u^{(+)} \\ \frac{\vec{\sigma} \cdot \vec{k}}{\varepsilon + m} \psi_u^{(+)} \end{pmatrix}$$

$\Rightarrow$ reduced a 4-component unknown spinor to 2 unknown components

Similarly $\psi^{(-)} = \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{k}}{\varepsilon + m} \psi_\ell^{(-)} \\ \psi_\ell^{(-)} \end{pmatrix}$.

Choose a basis: $\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and

Define

$$\boxed{u_r(\vec{k}) = \sqrt{\varepsilon + m} \begin{pmatrix} \chi_r \\ \frac{\vec{\sigma} \cdot \vec{k}}{\varepsilon + m} \chi_r \end{pmatrix}; \quad v_r(\vec{k}) = \sqrt{\varepsilon + m} \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{k}}{\varepsilon + m} \chi_r \\ \chi_r \end{pmatrix}} \quad r = 1, 2$$

then we write

$$\boxed{\psi(x) = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3\, 2E_k} \left\{ \hat{b}_{\vec{k},r}\, u_r(\vec{k})\, e^{-ik\cdot x} + \hat{d}^{+}_{\vec{k},r}\, v_r(\vec{k})\, e^{ik\cdot x} \right\}}$$

Canonical quantization: $\pi = \dfrac{\delta \mathcal{L}}{\delta(\partial_0 \psi)} = \bar{\psi}\, i\gamma^0 =$

$= \psi^+ \gamma_0 \gamma^0 \cdot i = i\psi^+$ as $\gamma_0\gamma^0 = \mathbb{1}$.

promote $\hat{b}$ & $\hat{d}$ to operators (note that $(i\gamma^\mu\partial_\mu - m)\psi = 0$ still holds!)

$\Rightarrow H = \int d^3x\, [\pi \dot{\psi} - \mathcal{L}] = \int d^3x \cdot [i\psi^+ \dot{\psi} - \mathcal{L}]$

$= \int d^3x\, [i\bar{\psi}\gamma^0 \partial_0 \psi - \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi] =$

$= \int d^3x\, [\cancel{i\bar{\psi}\gamma^0 \partial_0 \psi} - \cancel{\bar{\psi}\, i\gamma^0 \partial_0 \psi} - i\bar{\psi}\gamma^i \partial_i \psi +$

$+ \bar{\psi}\psi m] = \int d^3x\, \bar{\psi} \underbrace{[-i\gamma^i \partial_i + m]\psi}_{i\gamma^0 \partial_0 \psi \text{ (Dirac eqn.)}}$

$= \int d^3x\, i\psi^+ \partial_0 \psi \Rightarrow \boxed{H = \int d^3x\, i\psi^+ \partial_0 \psi}$

H is not ≥ 0 at the classical level ~ problem!

$\Rightarrow$ this is cured by quantization!

Plug in the solution of Dirac equation into the Hamiltonian:

$$\psi^+ = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \left\{ \hat{b}^+_{\vec{k},r}\, u^+_r(\vec{k})\, e^{ik\cdot x} + \hat{d}_{\vec{k},r}\, v^+_r(\vec{k})\, e^{-ik\cdot x} \right\}$$

$$\Rightarrow H = \int d^3x\; i\psi^+ \partial_0 \psi = \int d^3x \sum_{r,r'=1}^{2} \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \frac{d^3k'}{(2\pi)^3 2\varepsilon_{k'}}\, i \cdot$$

$$\cdot \left[\hat{b}^+_{\vec{k}',r'}\, u^+_{r'}(\vec{k}')\, e^{ik'\cdot x} + \hat{d}_{\vec{k}',r'}\, v^+_{r'}(\vec{k}')\, e^{-ik'\cdot x} \right] \cdot \left[\hat{b}_{\vec{k},r}\, u_r(\vec{k}) \cdot \right.$$

$$\left. \cdot (-i\varepsilon_k)\, e^{-ik\cdot x} + \hat{d}^+_{\vec{k},r}\, v_r(\vec{k})\, (i\varepsilon_k)\, e^{ik\cdot x} \right]$$

1) $\hat{b}^+\hat{b}$ -term: $\int d^3x\; e^{ik'\cdot x - ik\cdot x} = (2\pi)^3\, \delta^3(\vec{k}-\vec{k}')$

$$\Rightarrow \text{get} \quad \sum_{r,r'=1}^{2} \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \frac{d^3k'}{(2\pi)^3 2\varepsilon_k}\, \varepsilon_k\, (2\pi)^3 \delta(\vec{k}-\vec{k}')\, \hat{b}^+_{\vec{k}',r'}\, \hat{b}_{\vec{k},r}$$

$$u^+_{r'}(\vec{k}')\, u_r(\vec{k}) = \sum_{r,r'=1}^{2} \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k}\, \frac{1}{2}\, \hat{b}^+_{\vec{k},r'}\, \hat{b}_{\vec{k},r}\, u^+_{r'}(\vec{k})\, u_r(\vec{k})$$

$$\Rightarrow u_r(\vec{k}) = \sqrt{\varepsilon_k+m} \begin{pmatrix} \chi_r \\ \frac{\vec{\sigma}\cdot\vec{k}}{\varepsilon_k+m}\chi_r \end{pmatrix} \Rightarrow u^+_{r'} u_r = (\varepsilon_k+m)\left[\chi^T_{r'} \cdot \chi_r + \right.$$

$$\left. + \chi^T_{r'} \frac{\overbrace{\vec{\sigma}^+}^{=\vec{\sigma}} \cdot \vec{k}}{\varepsilon_k+m} \frac{\vec{\sigma}\cdot\vec{k}}{\varepsilon_k+m} \chi_r \right] = (\varepsilon_k+m)\left[\delta_{rr'} + \frac{\vec{k}^2}{(\varepsilon_k+m)^2}\delta_{rr'} \right]$$

$$= \delta_{rr'} \frac{1}{\varepsilon_k+m}\left[(\varepsilon_k+m)^2 + \overbrace{\varepsilon_k^2 - m^2}^{\vec{k}^2} \right] = \delta_{rr'} \frac{1}{\varepsilon_k+m}\left(2\varepsilon_k^2 + 2\varepsilon_k m\right) =$$

$$= 2\varepsilon_k \delta_{rr'}$$

$\Rightarrow$ $\hat b^\dagger \hat b$-term $= \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3 2E_k}\, E_k\, \hat b^\dagger_{\vec k,r}\, \hat b_{\vec k,r}$.

(2) $\hat b^\dagger \hat d^\dagger$ term: $\int d^3x\, e^{ik'\cdot x + ik\cdot x} = e^{2iE_k\cdot t}(2\pi)^3\delta(\vec k + \vec k')$

$\Rightarrow$ get $\propto u^\dagger_{r'}(-\vec k)\, v_r(\vec k)$

$$v_r(\vec k) = \sqrt{E_k+m}\begin{pmatrix} \frac{\vec\sigma\cdot\vec k}{E_k+m}\chi_r \\ \chi_r \end{pmatrix} \Rightarrow u^\dagger_{r'}(-\vec k)\, v_r(\vec k) =$$

$$= (E_k+m)\left(\chi^T_{r'} \quad \chi^T_{r'}\frac{\vec\sigma^\dagger\cdot(-\vec k)}{E_k+m}\right)\begin{pmatrix} \frac{\vec\sigma\cdot\vec k}{E_k+m}\chi_r \\ \chi_r \end{pmatrix} \propto$$

$$\propto \chi^T_{r'}(\vec\sigma\cdot\vec k)\chi_r - \chi^T_{r'}(\vec\sigma\cdot\vec k)\chi_r = 0$$

In the end get

$$\boxed{H = \sum_{r=1}^{2}\int\frac{d^3k}{(2\pi)^3\, 2E_k}\, E_k\left[\hat b^\dagger_{\vec k,r}\hat b_{\vec k,r} - \hat d_{\vec k,r}\hat d^\dagger_{\vec k,r}\right]}$$

Still not positive definite? Really, if we define some commutation relation for $\hat d, \hat d^\dagger$ $\Rightarrow$ would get $\hat b^\dagger\hat b - \hat d^\dagger\hat d$ ~ not good!

Define anti-commutation relations:

$$\{\hat{b}_{\vec{k},r}, \hat{b}^+_{\vec{k}',r'}\} = \{\hat{d}_{\vec{k},r}, \hat{d}^+_{\vec{k}',r'}\} = (2\pi)^3\, 2\mathcal{E}_k\, \delta_{rr'}\, \delta^3(\vec{k}-\vec{k}')$$

$$\{\hat{b}_{\vec{k},r}, \hat{b}_{\vec{k}',r'}\} = \{\hat{b}^+_{\vec{k},r}, \hat{b}^+_{\vec{k}',r'}\} = 0$$

$$\{\hat{d}_{\vec{k},r}, \hat{d}_{\vec{k}',r'}\} = \{\hat{d}^+_{\vec{k},r}, \hat{d}^+_{\vec{k}',r'}\} = 0$$

$\{\hat{b}, \hat{d}^+\} = 0$

$\{d, \hat{b}^+\} = 0$

$\{\hat{b}, \hat{d}\} = \{\hat{b}^+, \hat{d}^+\} = 0$

where $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ ~ anti-commutators

$\Rightarrow$ dropping ∞ number get

$$H = \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3\, 2\mathcal{E}_k}\, \mathcal{E}_k \left[\hat{b}^+_{\vec{k},r}\hat{b}_{\vec{k},r} + \hat{d}^+_{\vec{k},r} d_{\vec{k},r}\right]$$

Now it's positive-definite!

For the fields get

$$\{\psi_\alpha(\vec{x},t), \underbrace{\pi_\beta(\vec{x}',t)}_{= i\psi_\beta^+}\} = i\delta_{\alpha\beta}\,\delta(\vec{x}-\vec{x}')$$

$$\{\psi_\alpha, \psi_\beta\} = \{\psi^+_\alpha, \psi^+_\beta\} = 0$$

anti-commutation relations.

$\Rightarrow$ all operators anti-commute.

Time evolution:

$$+i\frac{\partial}{\partial t}\psi(x) = [\psi, H]$$

$$i\frac{\partial}{\partial t}\bar{\psi}(x) = [\bar{\psi}, H]$$

Still uses commutators (can show)

see HW

Useful formulas: $\bar{u}_r(\vec{k})\, u_s(\vec{k}) = 2m\, \delta_{rs}$

$$\bar{v}_r(\vec{k})\, v_s(\vec{k}) = -2m\, \delta_{rs}$$

$$u_r^{+}(\vec{k})\, u_s(\vec{k}) = 2\, \mathcal{E}_k\, \delta_{rs}$$

$$v_r^{+}(\vec{k})\, v_s(\vec{k}) = 2\, \mathcal{E}_k\, \delta_{rs}$$

$$\left.\begin{aligned} \sum_{r=1}^{2} u_{r,\alpha}(\vec{k})\, \bar{u}_{r,\beta}(\vec{k}) &= (\gamma\cdot k + m)_{\alpha\beta} \\ \sum_{r=1}^{2} v_r(\vec{k})\, \bar{v}_r(\vec{k}) &= \gamma\cdot k - m. \end{aligned}\right\}\ \text{can prove}$$

Conserved current of Dirac Lagrangian:

$$j^\mu = \bar\psi \gamma^\mu \psi \qquad \partial_\mu j^\mu = 0$$

Conserved charge is

$$Q = \int d^3x \, j^0(\vec{x},t) = \int d^3x \, \psi^+ \psi \overset{\text{can show}}{=} \sum_{r=1}^{2} \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \cdot$$

$$\cdot \left[\hat{b}^+_{\vec{k},r} \, \hat{b}_{\vec{k},r} - \hat{d}^+_{\vec{k},r} \, \hat{d}_{\vec{k},r} \right]$$

$\Rightarrow$ just like for complex scalar fields, we see that

$\hat{b}^+_{\vec{k},r}$ ~ creates particles (charge +1)

$\hat{d}^+_{\vec{k},r}$ ~ creates anti-particles (charge −1).

both particles & anti-particles could be helicity + or − $\Rightarrow$ 2 d.o.f. each ($r = \pm 1$).

Fock states: $\hat{b}^+_{\vec{k},r} \, \hat{d}^+_{\vec{p},r'} |0\rangle$, ...

Note: can not have 2 identical particles:

$$\hat{b}^+_{\vec{k},r} \, \hat{b}^+_{\vec{k},r} |0\rangle = 0 \quad \text{as} \left(\hat{b}^+_{\vec{k},r} \right)^2 = 0$$

(anti-commutation relations)

$\Rightarrow$ Pauli exclusion principle works!