Last time: defined Dirac spinors $u_r(\vec{k})$ & $v_r(\vec{k})$:

$$\boxed{u_r(\vec{k}) = \sqrt{\mathcal{E}_k+m}\begin{pmatrix} \chi_r \\ \frac{\vec{\sigma}\cdot\vec{k}}{\mathcal{E}_k+m}\chi_r \end{pmatrix}, \quad v_r(\vec{k}) = \sqrt{\mathcal{E}_k+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{\mathcal{E}_k+m}\chi_r \\ \chi_r \end{pmatrix}.}$$

$r = 1, 2$

$$\chi_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad \chi_2 = \begin{pmatrix}0\\1\end{pmatrix}.$$

Solution of Dirac equation. General solution is

$$\boxed{\psi(x) = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \sum_{r=1}^{2} \left\{ \hat{b}_{\vec{k},r}\, u_r(\vec{k}) e^{-ik\cdot x} + \hat{d}^{+}_{\vec{k},r}\, v_r(\vec{k})\, e^{ik\cdot x} \right\}}$$

Quantized free Dirac field: $\hat{b}$, $\hat{d}^+$ ~ operators.

Showed that for $\hat{H}$ to give energy ≥ 0 need anti-commutation relations:

$$\boxed{\left\{ \hat{b}_{\vec{k},r}, \hat{b}^{+}_{\vec{k}',r'} \right\} = \left\{ \hat{d}_{\vec{k},r}, \hat{d}^{+}_{\vec{k}',r'} \right\} = (2\pi)^3\, 2\mathcal{E}_k\, \delta_{rr'}\, \delta^3(\vec{k}-\vec{k}')}$$

(all other anti-commutators are $\emptyset$.)

Then
$$\boxed{H = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \sum_{r=1}^{2} \mathcal{E}_k \left[\hat{b}^{+}_{\vec{k},r}\hat{b}_{\vec{k},r} + \hat{d}^{+}_{\vec{k},r}\hat{d}_{\vec{k},r} \right]}.$$

Canonical equal-time anti-commutation relations:

$\alpha,\beta = 1,2,3,4$ Dirac indices

$$\left\{ \psi_\alpha(\vec{x},t), \pi_\beta(\vec{y},t) \right\} = i\delta_{\alpha\beta}\, \delta(\vec{x}-\vec{y})$$

$$\left\{ \psi_\alpha(\vec{x},t), \psi_\beta(\vec{y},t) \right\} = \left\{ \psi^{+}_\alpha(\vec{x},t), \psi^{+}_\beta(\vec{y},t) \right\} = 0$$

Conserved $U(1)$ current: $j^\mu = \bar{\psi}\gamma^\mu\psi \, , \; \partial^\mu j_\mu = 0$

Conserved charge is

$$Q = \int \frac{d^3k}{(2\pi)^3 2E_k} \sum_{r=1}^{2} \left[\hat{b}^+_{\vec{k},r} \hat{b}_{\vec{k},r} - \hat{d}^+_{\vec{k},r} \hat{d}_{\vec{k},r} \right]$$

$\hat{b}^+$ ~ creates particles

$\hat{d}^+$ ~ creates anti-particles

Fock states: $\hat{b}^+_{\vec{k}_1,r_1} \hat{b}^+_{\vec{k}_2,r_2} \hat{d}^+_{\vec{k}_3,r_3} \ldots |0\rangle$

$\left(\hat{b}^+_{\vec{k},r}\right)^2 |0\rangle = 0$ (Pauli exclusion principle)

Quantization of Vector Field A_μ

Lagrangian and EOM.

Consider a vector field $A_\mu(x)$. What is its Lagrangian?

Start with Dirac field $\psi(x)$ with Lagrangian

$$\mathcal{L} = \bar\psi\,[\,i\gamma^\mu\partial_\mu - m\,]\,\psi.$$

This Lagrangian is invariant under

$$\begin{cases} \psi(x) \to e^{i\alpha}\psi(x) \\ \bar\psi(x) \to e^{-i\alpha}\bar\psi(x). \end{cases} \qquad \text{(global symmetry)}$$

Global symmetry: α is independent of x^μ, the same transformation for all points in space.

Let's make it a local symmetry: we want the Lagrangian to be symmetric under

$$\begin{cases} \psi(x) \to e^{i\alpha(x)}\psi(x) \\ \bar\psi(x) \to e^{-i\alpha(x)}\bar\psi(x). \end{cases} \qquad \text{(local symmetry)}$$

where $\alpha(x)$ is a (real-valued) function now.

What happens to $\mathcal{L}$?

$$\bar\psi\,[\,i\gamma^\mu\partial_\mu - m\,]\,\psi \to \bar\psi\, e^{-i\alpha(x)}[\,i\gamma^\mu\partial_\mu - m\,]\,e^{i\alpha(x)}\psi =$$

$$= \bar{\psi}\left[i\gamma^\mu\partial_\mu - m\right]\psi - \bar{\psi}\gamma^\mu(\partial_\mu\alpha)\psi$$

$\Rightarrow \mathcal{L}_{Dirac}$ is not invariant under local $U(1)$ symm!

$\Rightarrow$ Fix this by introducing gauge field A_μ:

$$\mathcal{L} = \bar{\psi}\left[i\gamma^\mu D_\mu - m\right]\psi$$

where

Def. $D_\mu = \partial_\mu + ieA_\mu$ is the <u>covariant derivative</u>.

$$\mathcal{L}_1 = \bar{\psi}\left[i\gamma^\mu\partial_\mu - m\right]\psi - e\bar{\psi}\gamma^\mu\psi A_\mu$$

(Remember in E&M: $\mathcal{L}_{int.} = -j^\mu A_\mu$ ~ interaction Lagrangian

$\Rightarrow$ in Dirac theory conserved $U(1)$ current

is $j^\mu = e\bar{\psi}\gamma^\mu\psi \Rightarrow \mathcal{L}_{int} = -j^\mu A_\mu = -e\bar{\psi}\gamma^\mu\psi A_\mu$.

~ we have not done anything new compared to E&M.)

Demand that the $U(1)$ local transformation is

$$\begin{cases} \psi(x) \to e^{i\alpha(x)}\psi(x), \quad \bar{\psi} \to e^{-i\alpha}\bar{\psi}. \\ A_\mu(x) \to A_\mu - \frac{1}{e}\partial_\mu\alpha. \end{cases}$$

Then

$$\mathcal{L}_1 \to \bar{\psi}\left[i\gamma^\mu\partial_\mu - m\right]\psi - \cancel{\bar{\psi}\gamma^\mu(\partial_\mu\alpha)\psi} - e\bar{\psi}\gamma^\mu\psi \cdot \left(A_\mu - \cancel{\tfrac{1}{e}\partial_\mu\alpha}\right) = \mathcal{L}_1$$

$\Rightarrow$ now $\mathcal{L}_1$ is invariant!

However, we need a Lagrangian for A_μ field itself! We impose usual requirements (like for scalar field φ): at most quadratic in A_μ, ∂_μ. On top of that require that $\mathcal{L}_{A_\mu}$ is gauge-invariant, i.e., invariant under

$$A_\mu \to A_\mu - \frac{1}{e} \partial_\mu \alpha .$$

Start by constucting a gauge-invariant field strength tensor $F_{\mu\nu}$:

Def. $$\boxed{F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu}$$ (field strength tensor)

There is also dual field strength tensor

$$\boxed{\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}}$$

$$\varepsilon^{0123} = +1 \ , \quad \varepsilon^{\mu\nu\rho\sigma} = -\varepsilon^{\nu\mu\rho\sigma}, \ \ldots$$

Lorentz-invariants are:

$$F_{\mu\nu} F^{\mu\nu} \ , \quad F_{\mu\nu} \tilde{F}^{\mu\nu} \ , \quad \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu}$$

However, $F_{\mu\nu} F^{\mu\nu} = - \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu}$ => down to two.

Now, one can easily show that $F_{\mu\nu} \tilde{F}^{\mu\nu} = \partial_\mu K^\mu$

with some 4-vector K^μ. Hence

$$\int d^4x \; F_{\mu\nu} \tilde{F}^{\mu\nu} = \int d^4x \; \partial_\mu K^\mu = \int\limits_{\substack{\text{surface} \\ \text{(3dim)}}} d\sigma_\mu \; K^\mu \quad \Rightarrow$$

$\Rightarrow$ this is the surface term, does not give any classical dynamics $\Rightarrow$ no good to use $F_{\mu\nu}\tilde{F}^{\mu\nu}$ for Lagrangian.

$\Rightarrow$ only $F_{\mu\nu} F^{\mu\nu}$ is left.

Matching the constants on E&M get

$$\mathcal{L}_{gauge} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

The full Lagrangian for spinors (electrons) interacting with gauge field (photons) is

$$\mathcal{L}_{QED} = \bar{\psi}\left[i\gamma^\mu D_\mu - m \right]\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

QED = Quantum ElectroDynamics.

$$\mathcal{L}_{QED} = \underbrace{\underbrace{\bar{\psi}\left[i\gamma^\mu \partial_\mu - m \right]\psi}_{\text{Dirac}} - \underbrace{\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{\text{gauge}}}_{\text{free field terms}} \underbrace{- e\,\bar{\psi}\gamma^\mu\psi A_\mu}_{\substack{\text{interaction term} \\ \text{(consider later)}}}$$

Take $\mathcal{L}_{E\&M} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$.

Let's find its equations of motion. Euler-Lagrange equations are:

$$\frac{\delta \mathcal{L}}{\delta A_\mu} - \partial_\nu \left[\frac{\delta \mathcal{L}}{\delta(\partial_\nu A_\mu)} \right] = 0.$$

$$\frac{\delta \mathcal{L}}{\delta A_\mu} = -j^\mu; \quad \frac{\delta \mathcal{L}}{\delta(\partial_\nu A_\mu)} = -\frac{1}{4} \frac{\delta (F_{\alpha\beta} F^{\alpha\beta})}{\delta(\partial_\nu A_\mu)} =$$

$$= -\frac{1}{4} g^{\alpha\rho} g^{\beta\sigma} \frac{\delta(F_{\alpha\beta} F_{\rho\sigma})}{\delta(\partial_\nu A_\mu)} = -\frac{1}{4} g^{\alpha\rho} g^{\beta\sigma} \Big[(\delta_\alpha^\nu \delta_\beta^\mu -$$

$$- \delta_\alpha^\mu \delta_\beta^\nu) F_{\rho\sigma} + F_{\alpha\beta} (\delta_\rho^\nu \delta_\sigma^\mu - \delta_\rho^\mu \delta_\sigma^\nu) \Big] =$$

$$= -\frac{1}{4} \left[F^{\nu\mu} - F^{\mu\nu} + F^{\nu\mu} - F^{\mu\nu} \right] = F^{\mu\nu}.$$

$$\Rightarrow \frac{\delta \mathcal{L}}{\delta(\partial_\nu A_\mu)} = F^{\mu\nu}$$

$\Rightarrow$ EOM are $-j^\mu - \partial_\nu F^{\mu\nu} = 0$

$$\Rightarrow \boxed{\partial_\nu F^{\nu\mu} = j^\mu}$$ Maxwell equations (as expected)

$$F_{\mu\nu} = [D_\mu D_\nu - D_\nu D_\mu] \frac{-i}{e}$$ (useful formula)

$$\partial_\nu F^{\nu\mu} = j^\mu \Rightarrow \partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu) = j^\mu$$

$$\Rightarrow \Box A^\mu - \partial^\mu \partial_\nu A^\nu = j^\mu$$

Canonical Quantization of free vector field ($j^\mu = 0$)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Two ways to do canonical quantization:

(II) in Lorenz gauge $\partial_\mu A^\mu = 0 \Rightarrow \Box A^\mu = 0$

Now, Lorenz gauge has gauge sub-freedom:

$A^\mu \to A'^\mu = A^\mu + \partial^\mu \Lambda \Rightarrow$ want $\partial_\mu A'^\mu = 0$ too

$\Rightarrow$ if $\partial_\mu A^\mu = 0 \Rightarrow$ need $\Box \Lambda = 0$ $\Rightarrow$ for any such Λ still have $\partial_\mu A'^\mu = 0$.

Choose Λ such that $A^0 = -\partial^0 \Lambda \Rightarrow$

$\Rightarrow A'^0 = 0$, $\vec{\nabla}\cdot\vec{A}' = 0$ $\Rightarrow$ can work

(I) in Coulomb gauge $A^0 = 0$, $\vec{\nabla}\cdot\vec{A} = 0$
(radiation)

Can one always pick this gauge? $\Lambda = -\frac{1}{\partial^0} A^0$

$\Rightarrow$ plug back into $\Box \Lambda = 0 \Rightarrow ((\partial_0)^2 - \vec{\nabla}^2)\Lambda = 0 \Rightarrow$

$\Rightarrow -\partial^0 A^0 = \vec{\nabla}^2 \Lambda \Rightarrow \Lambda = -\frac{\partial^0}{\vec{\nabla}^2} A^0 = -\frac{1}{\partial_0} A^0$

$\Rightarrow [(\partial^0)^2 - \vec{\nabla}^2] A^0 = 0 \Rightarrow \Box A^0 = 0$ ~ 0^{th} component of maxwell equations $\Rightarrow$ valid!