Last time: Quantization of Vector Field A_μ (cont'd)

Lagrangian and EOM

$\mathcal{L}_{Dirac} = \bar{\psi}[i\gamma^\mu \partial_\mu - m]\psi$, symmetric under $U(1)$ global:

$$\psi \to e^{i\alpha}\psi, \quad \bar{\psi} \to e^{-i\alpha}\bar{\psi}$$

Make $U(1)$ local : $\psi \to e^{i\alpha(x)}\psi, \quad \bar{\psi} \to e^{-i\alpha(x)}\bar{\psi}$

$\Rightarrow$ new Lagrangian invariant under $U(1)$ local is

$$\mathcal{L}_1 = \bar{\psi}[i\gamma^\mu D_\mu - m]\psi$$

where

Def. $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative

$$A_\mu \xrightarrow{U(1)} A_\mu - \frac{1}{e}\partial_\mu \alpha .$$

The Lagrangian for A_μ can be constructed by requiring it to be gauge-inv. & quadratic in A_μ, ∂_μ.

$$\mathcal{L}_{gauge} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ gauge field

$$\boxed{\mathcal{L}_{QED} = \bar{\psi}[i\gamma^\mu D_\mu - m]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}$$

$$\mathcal{L}_{E\&M} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu \Rightarrow \boxed{\partial_\nu F^{\nu\mu} = j^\mu}$$

Maxwell equations

Canonical Quantization of Free Vector Field

(cont'd)

Put $j^\mu = 0$ (free field) $\Rightarrow$ $\partial_\nu F^{\nu\mu} = 0$

$$\Rightarrow \boxed{\Box A^\mu - \partial^\mu \partial_\nu A^\nu = 0}$$ Maxwell equations in vacuum

(II) Lorenz gauge $\partial_\mu A^\mu = 0$

(I) Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$, $A^0 = 0$ (in vacuum only)

$$\partial_\nu F^{\nu\mu} = j^\mu \Rightarrow \partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu) = j^\mu$$

$$\Rightarrow \boxed{\Box A^\mu - \partial^\mu \partial_\nu A^\nu = j^\mu}$$

Canonical Quantization of free vector field ($j^\mu = 0$)

$$\boxed{\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}}$$

Two ways to do canonical quantization:

(II) in Lorenz gauge $\partial_\mu A^\mu = 0 \Rightarrow \Box A^\mu = 0$

Now, Lorenz gauge has gauge sub-freedom:

$A^\mu \to A'^\mu = A^\mu + \partial^\mu \Lambda \Rightarrow$ want $\partial_\mu A'^\mu = 0$ too

$\Rightarrow$ if $\partial_\mu A^\mu = 0 \Rightarrow$ need $\Box \Lambda = 0 \Rightarrow$ for any such Λ still have $\partial_\mu A'^\mu = 0$.

Choose Λ such that $A^0 = -\partial^0 \Lambda \Rightarrow$

$\Rightarrow A'^0 = 0$, $\vec{\nabla}\cdot\vec{A}' = 0 \Rightarrow$ can work

(I) in Coulomb gauge $A^0 = 0$, $\vec{\nabla}\cdot\vec{A} = 0$
(radiation)

Can one always pick this gauge? $\Lambda = -\frac{1}{\partial^0} A^0$

$\Rightarrow$ plug back into $\Box\Lambda = 0 \Rightarrow ((\partial_0)^2 - \vec{\nabla}^2)\Lambda = 0 \Rightarrow$

$\Rightarrow -\partial^0 A^0 = \vec{\nabla}^2 \Lambda \Rightarrow \Lambda = -\frac{\partial^0}{\vec{\nabla}^2} A^0 = -\frac{1}{\partial_0} A^0$

$\Rightarrow \left[(\partial^0)^2 - \vec{\nabla}^2\right] A^0 = 0 \Rightarrow \Box A^0 = 0$ ~ 0^{th} component of maxwell equations $\Rightarrow$ valid!

(I) Coulomb gauge quantization

$$A^0 = 0, \quad \vec{\nabla}\cdot\vec{A} = 0$$

$\pi_i = \dfrac{\delta \mathcal{L}}{\delta \dot{A}^i}$ ~ canonical momenta, A^i ~ free fields (A^0 is fixed)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$\pi^i = \frac{\delta \mathcal{L}}{\delta \dot{A}_i} = \frac{\delta \mathcal{L}}{\delta(\partial_0 A_i)} \underset{\substack{\uparrow \\ \text{see above.}}}{=} -F^{0i} = E^i \quad \text{(electric field)}$$

$$\text{''} \; -\dot{A}^i = \dot{A}_i$$

$\Rightarrow$ E^i is the canonical momentum!

(conjugate to A_i ~ lower "i")

$\Rightarrow$ canonical commutation relation could be:

$$[A_i(\vec{x},t), \pi^j(\vec{y},t)] \overset{?}{=} i\delta_i{}^j \, \delta(\vec{x}-\vec{y})$$

but: this would violate gauge condition $\vec{\nabla}\cdot\vec{A} = 0$

as, if we act with $\vec{\nabla}_x$ on it we get

$$[\underbrace{-\vec{\nabla}\cdot\vec{A}(\vec{x},t)}_{\overset{\shortparallel}{0}}, \pi^j(\vec{y},t)] = i\delta_i{}^j \vec{\nabla}_x \delta(\vec{x}-\vec{y}) \neq 0$$

$\Rightarrow$ replace $\delta_i{}^j$ with $d_i{}^j$ such that $d_i{}^j \nabla^i = 0$

$\Rightarrow$ $d_i{}^j = \delta_i{}^j + \dfrac{\partial_i \partial^j}{\vec{\nabla}^2}$ clearly works ($\partial^i = -\partial_i = \nabla^i$)

$$\left(\Rightarrow d_{ij} = g_{ij} + \frac{\partial_i \partial_j}{\vec{\nabla}^2} = -\delta_{ij} + \frac{\partial_j \partial_i}{\vec{\nabla}^2}\right)$$

$$\Rightarrow \boxed{\begin{aligned} &\left[A_i(\vec{x},t),\ \pi^j(\vec{y},t) \right] = i\left(\delta_i{}^j + \frac{\partial_i \partial^j}{\vec{\nabla}^2} \right) \delta(\vec{x}-\vec{y}) \\ &\left[A_i(\vec{x},t),\ A_j(\vec{y},t) \right] = \left[\pi^i(\vec{x},t),\ \pi^j(\vec{y},t) \right] = 0 \end{aligned}}$$

correct Coulomb gauge commutation relations
($[A^i, \pi^j] = i\left(g^{ij} + \frac{\partial^i\partial^j}{\vec{\partial}^2}\right)\delta(\vec{x}-\vec{y}) = -i\left(\delta^{ij} - \frac{\partial^i\partial^j}{\vec{\partial}^2}\right)\delta(\vec{x}-\vec{y})$).

Maxwell equations $\quad \Box A^\mu = 0 \Rightarrow \Box A^i = 0$

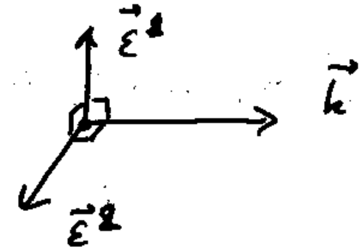
$$\Rightarrow \vec{A}(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k} \sum_{\lambda=1}^{2} \vec{\varepsilon}^{\,\lambda}(\vec{k}) \left[\hat{a}_{\vec{k},\lambda}\, e^{-ik\cdot x} + \hat{a}^+_{\vec{k},\lambda}\, e^{ik\cdot x} \right]$$

Now $\varepsilon_k = |\vec{k}|$ as there is no mass.

$\vec{\varepsilon}^{\,\lambda}(\vec{k})$ ~ polarization vectors, $\lambda = 1, 2$.

Impose gauge condition $\vec{\nabla}\cdot\vec{A} = 0 \Rightarrow$

$\Rightarrow \vec{k}\cdot\vec{\varepsilon}^{\,\lambda}(\vec{k}) = 0 \quad \Rightarrow$



choose an orthonormal basis

for $\vec{\varepsilon}^{\,\lambda}$'s as shown here ↗

$$\vec{\varepsilon}^{\,\lambda}(\vec{k})\cdot\vec{\varepsilon}^{\,\lambda'}(\vec{k}) = \delta^{\lambda\lambda'}$$

$$\boxed{\begin{aligned} &\left[\hat{a}_{\vec{k},\lambda},\ \hat{a}^+_{\vec{k}',\lambda'} \right] = (2\pi)^3\, 2\varepsilon_k\, \delta_{\lambda\lambda'}\, \delta^3(\vec{k}-\vec{k}') \\ &\left[\hat{a}_{\vec{k},\lambda},\ \hat{a}_{\vec{k}',\lambda'} \right] = \left[\hat{a}^+_{\vec{k},\lambda},\ \hat{a}^+_{\vec{k}',\lambda'} \right] = 0 \end{aligned}}$$

commutation relations for creation & annihilation operators

The Hamiltonian is then

$$H = \int d^3x \, [\underset{\dot{A}_i}{\underset{\parallel}{\pi^i}} \dot{A}_i - \mathcal{L}] = \int d^3x \, \Big[(\dot{A}_i)^2 - \left(-\tfrac{1}{4}\right)\big[2 \underbrace{F_{0i}}_{\dot{A}_i} \underbrace{F^{0i}}_{-\dot{A}_i}$$

$$+ F_{ij} F^{ij} \big] \Big] = \int d^3x \Big[(\dot{A}_i)^2 - \tfrac{1}{2}(\dot{A}_i)^2 + \tfrac{1}{4}(F_{ij})^2 \Big]$$

$$= \int d^3x \Big[\tfrac{1}{2}(\dot{A}_i)^2 + \tfrac{1}{4}\overbrace{(F_{ij})^2}^{2\vec{B}^2} \Big] \quad , \quad \vec{B} \sim \text{magnetic field}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}.$$

$$\Rightarrow \quad H = \frac{1}{2} \int d^3x \, [\vec{E}^2 + \vec{B}^2] \quad \Rightarrow \text{energy} \geq 0$$

(good.)

Plugging in the $\vec{A}$-field we get (after some math)

$$\boxed{H = \int \frac{d^3k}{(2\pi)^3 \, 2\varepsilon_k} \sum_{\lambda=1}^{2} \varepsilon_k \, \hat{a}^{+}_{\vec{k},\lambda} \, \hat{a}_{\vec{k},\lambda}}$$

$\hat{a}^{+}_{\vec{k},\lambda}$ ~ creation operator

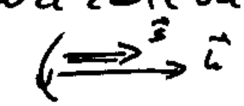
$\hat{a}_{\vec{k},\lambda}$ ~ annihilation operator

$\lambda = 1, 2$ $\Rightarrow$ photons have 2 degrees of freedom (2 polarizations)

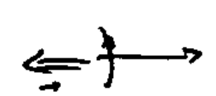
$\vec{\varepsilon}^{\,1}, \vec{\varepsilon}^{\,2}$ $\Rightarrow$ can choose circular polarizations

$$\vec{\varepsilon}^{\,\pm} = \frac{\vec{\varepsilon}^{\,1} \pm i \vec{\varepsilon}^{\,2}}{\sqrt{2}}$$

$+$ ~ helicity $+1$



$-$ ~ helicity -1.



(II) Lorenz gauge quantization

in Coulomb gauge physics is still Lorentz-invariant, but not manifestly so.

$\Rightarrow$ let's work in Lorenz gauge $\partial_\mu A^\mu = 0$

$A^\mu(x)$ ~ our fields

$$\pi^\mu = \frac{\delta \mathcal{L}}{\delta \dot{A}_\mu} = \frac{\delta \mathcal{L}}{\delta(\partial_0 A_\mu)} = F^{\mu 0} = -F^{0\mu}$$ ~ canonical momenta.

Problem: $\pi^0 = -F^{00} = \emptyset$. No π^0? Bad, since would not have commutation relation

$$[A_0(\vec{x},t), \pi_0(\vec{y},t)] = i g_{00}\, \delta(\vec{x}-\vec{y}). \quad \dots$$

Modify the Lagrangian to

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{2}(\partial_\mu A^\mu)^2$$ (not gauge invariant)

old term ↓ ↑ gauge-fixing term

$$\Rightarrow \frac{\delta \mathcal{L}}{\delta(\partial_\nu A_\mu)} = F^{\mu\nu} - \frac{\lambda}{2} \cdot \frac{\delta(g^{\alpha\beta}\partial_\alpha A_\beta\, g^{\sigma\rho}\partial_\sigma A_\rho)}{\delta(\partial_\nu A_\mu)} =$$

$$= F^{\mu\nu} - \lambda \underbrace{\underbrace{g^{\alpha\beta} g_\alpha{}^\nu}_{g^{\beta\nu}} g_\beta{}^\mu}_{g^{\mu\nu}} (\partial_\rho A^\rho) = F^{\mu\nu} - \lambda g^{\mu\nu}(\partial_\rho A^\rho)$$

$$\frac{\delta \mathcal{L}}{\delta A_\mu} = 0 \quad \text{still.}$$

$$\partial_\nu \left[\frac{\delta \mathcal{L}}{\delta(\partial_\nu A_\mu)} \right] = 0 \Rightarrow \partial_\nu F^{\mu\nu} - \lambda \partial^\mu \partial_\rho A^\rho = 0$$

$$\Rightarrow \partial_\nu \partial^\mu A^\nu - \Box A^\mu - \lambda \partial^\mu \partial_\rho A^\rho = 0$$

$$\Box A^\mu - (1-\lambda) \partial^\mu \partial_\nu A^\nu = 0$$

$\lambda = 1 \Rightarrow$ "Feynman gauge" $\Rightarrow$ get $\Box A^\mu = 0$ back.

$\Rightarrow$ the same classical physics with a different Lagrangian: $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2$.

Now, $\pi^0 = \dfrac{\delta \mathcal{L}}{\delta \dot{A}_0} = -\partial_\mu A^\mu$. No longer need to impose $\partial_\mu A^\mu = 0$ explicitly $\Rightarrow$ π^0 does not have to be $\emptyset$.

Quantize the system:

$$[A_\mu(\vec{x},t), \pi_\nu(\vec{y},t)] = i g_{\mu\nu} \delta(\vec{x}-\vec{y})$$

$$[A_\mu(\vec{x},t), A_\nu(\vec{y},t)] = [\pi_\mu(\vec{x},t), \pi_\nu(\vec{y},t)] = 0$$

$$A_\mu(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \sum_{\lambda=0}^{3} \varepsilon_\mu^\lambda(\vec{k}) \left[\hat{a}_{\vec{k},\lambda} e^{-ik\cdot x} + \hat{a}^+_{\vec{k},\lambda} e^{ik\cdot x} \right]$$

choose the basis of polarization vectors in such a way that $\varepsilon^\lambda \cdot \varepsilon^{\lambda'} = g^{\lambda\lambda'}$

One then gets:

$$\left[\hat{a}_{\vec{k},\lambda}\,,\ \hat{a}^{+}_{\vec{k}',\lambda'}\right] = -\,g_{\lambda\lambda'}\,(2\pi)^3\,2\varepsilon_k\,\delta(\vec{k}-\vec{k}')$$

all other commutators are zero.

if $k^\mu = (k, 0, 0, k) \Rightarrow$

$$\varepsilon^0 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad \varepsilon^1 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad \varepsilon^2 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad \varepsilon^3 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$\varepsilon^{1,2}\cdot k = 0$ (transverse)

$\varepsilon^3 \sim$ longitudinal, $\varepsilon^0 \sim$ time-like.

Fock space: $\hat{a}^{+}_{\vec{k},\lambda}|0\rangle, \ldots$

$$|1,\lambda\rangle = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} f(\vec{k})\,\hat{a}^{+}_{\vec{k},\lambda}|0\rangle \quad \sim \text{one-photon state}$$

<u>Problem</u>: $\langle 1, \lambda=0 | 1, \lambda=0\rangle = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k}\,\frac{d^3k'}{(2\pi)^3 2\varepsilon_{k'}}\cdot f(\vec{k})f^*(\vec{k}')$

$$\cdot\langle 0|\underbrace{\hat{a}_{\vec{k}',\lambda=0}\ \hat{a}^{+}_{\vec{k},0}}|0\rangle = -\int\frac{d^3k}{(2\pi)^3\,2\varepsilon_k}\,|f(\vec{k})|^2\,\langle 0|0\rangle$$

$$'' \quad \left[\hat{a}_{\vec{k}',0}\,,\ \hat{a}^{+}_{\vec{k},0}\right] = -(2\pi)^3\,2\varepsilon_k\,\delta(\vec{k}-\vec{k}')$$

$\Rightarrow$ negative norm states!

$$H = \int\frac{d^3k}{(2\pi)^3 2\varepsilon_k}\,\varepsilon_k\left[\sum_{\lambda=1}^{3}\hat{a}^{+}_{\vec{k},\lambda}\,\hat{a}_{\vec{k},\lambda} - \hat{a}^{+}_{\vec{k},0}\,\hat{a}_{\vec{k}0}\right] \sim \text{negative energy problem} \ldots$$