Last time:

Canonical Quantization of Free Vector Field (cont'd)

(I) Coulomb gauge: $\vec{\nabla}\cdot\vec{A} = 0$, $A^0 = 0$ (vacuum)

$\Rightarrow A^i(x) \sim$ free fields

$$\pi^i = \frac{\delta \mathcal{L}}{\delta \dot{A}_i} = F^{i0} = E^i \sim \text{canonical momenta}$$

$$\left[A_i(\vec{x},t), \pi^j(\vec{y},t)\right] = i\left(\delta_i{}^j + \frac{\partial_i \partial^j}{\vec{\nabla}^2}\right)\delta(\vec{x}-\vec{y})$$

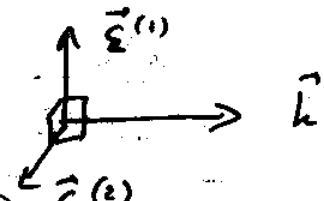
$$\left[A_i(\vec{x},t), A_j(\vec{y},t)\right] = 0 = \left[\pi^i(\vec{x},t), \pi^j(\vec{y},t)\right]$$

$\Box A^i = 0 \sim$ Maxwell eqn's

$$\vec{A}(x) = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \sum_{\lambda=1}^{2} \vec{\varepsilon}^{(\lambda)}(\vec{k})\left[\hat{a}_{\vec{k},\lambda} e^{-ik\cdot x} + \hat{a}^{+}_{\vec{k},\lambda} e^{ik\cdot x}\right]$$

Choose polarizations basis: $\vec{k}\cdot\vec{\varepsilon}^{(\lambda)}(\vec{k}) = 0$ (Coulomb gauge)

$$\vec{\varepsilon}^{(\lambda)} \cdot \vec{\varepsilon}^{(\lambda')} = \delta^{\lambda\lambda'}$$



$$\left[\hat{a}_{\vec{k},\lambda}, \hat{a}^{+}_{\vec{k}',\lambda'}\right] = (2\pi)^3 2\varepsilon_k \delta_{\lambda\lambda'} \delta(\vec{k}-\vec{k}')$$

$$\left[\hat{a}_{\vec{k},\lambda}, \hat{a}_{\vec{k}',\lambda'}\right] = \left[\hat{a}^{+}_{\vec{k},\lambda}, \hat{a}^{+}_{\vec{k}',\lambda'}\right] = 0.$$

$\lambda = 1, 2 \Rightarrow$ only 2 physical polarizations

$$H = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \sum_{\lambda=1}^{2} \varepsilon_k \hat{a}^{+}_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda}$$

(II) Lorenz gauge: $\partial_\mu A^\mu = 0$ $\Rightarrow$ $A^\mu(x)$ free fields

$\Rightarrow \pi^\mu = \frac{\delta \mathcal{L}}{\delta \dot{A}_\mu} = F^{\mu 0} \Rightarrow \pi^0 = 0$ ~ bad

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{2} (\partial_\mu A^\mu)^2$$ ~ new Lagrangian

EOM: $$\Box A^\mu - (1-\lambda)\, \partial^\mu \partial_\nu A^\nu = 0$$

$\lambda = 1 \rightarrow$ Feynman gauge $\Rightarrow$ $\Box A^\mu = 0$ is back.

$\pi^0 = -\partial_\mu A^\mu$ ~ non-zero (don't fix any gauge)

$$\Rightarrow [A_\mu(\vec{x}, t), \pi_\nu(\vec{y}, t)] = i g_{\mu\nu} \delta(\vec{x} - \vec{y})$$

$$[A_\mu(\vec{x}, t), A_\nu(\vec{y}, t)] = [\pi_\mu(\vec{x}, t), \pi_\nu(\vec{y}, t)] = 0$$

$\Box A^\mu = 0 \Rightarrow$

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k} \sum_{\lambda=0}^{3} \varepsilon_\mu^{(\lambda)}(\vec{k}) \left[\hat{a}_{\vec{k},\lambda} e^{-ik\cdot x} + \hat{a}^\dagger_{\vec{k},\lambda} e^{ik\cdot x} \right]$$

$\varepsilon^{(\lambda)} \cdot k \neq 0 \Rightarrow$ for $k^\mu = (k, 0, 0, k)$ have

$\varepsilon^{(0)} = (1, 0, 0, 0)$, $\varepsilon^{(1)} = (0, 1, 0, 0)$, $\varepsilon^{(2)} = (0, 0, 1, 0)$
time-like

$\varepsilon^{(3)} = (0, 0, 0, 1)$, $\varepsilon^{(1,2)} \cdot k = 0$ (transverse)
longitudinal

$$[\hat{a}_{\vec{k},\lambda}, \hat{a}^\dagger_{\vec{k}',\lambda'}] = -g_{\lambda\lambda'} (2\pi)^3\, 2\varepsilon_k\, \delta(\vec{k} - \vec{k}')$$ (all others zero)

One then gets:

$$\boxed{\left[\hat{a}_{\vec{k},\lambda}\,,\ \hat{a}^{+}_{\vec{k}',\lambda'}\right] = -\,g_{\lambda\lambda'}\,(2\pi)^3\,2\varepsilon_k\,\delta(\vec{k}-\vec{k}')}$$

all other commutators are zero.

if $k^\mu = (k, 0, 0, k) \Rightarrow$

$$\varepsilon^0 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad \varepsilon^1 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad \varepsilon^2 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad \varepsilon^3 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$\varepsilon^{1,2}\cdot k = 0$ (transverse)

$\varepsilon^3 \sim$ longitudinal, $\varepsilon^0 \sim$ time-like.

Fock space: $\hat{a}^{+}_{\vec{k},\lambda}|0\rangle, \ldots$

$$|1,\lambda\rangle = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} f(\vec{k})\,\hat{a}^{+}_{\vec{k},\lambda}|0\rangle \quad \sim \text{one-photon state}$$

Problem: $\langle 1, \lambda=0 | 1, \lambda=0\rangle = \displaystyle\int \frac{d^3k}{(2\pi)^3 2\varepsilon_k}\frac{d^3k'}{(2\pi)^3 2\varepsilon_{k'}}\, f(\vec{k})\, f^*(\vec{k}')$

$$\langle 0 | \underbrace{\hat{a}_{\vec{k}',\lambda=0}\ \hat{a}^{+}_{\vec{k},0}}_{} |0\rangle = -\int \frac{d^3k}{(2\pi)^3 2\varepsilon_k}\,|f(\vec{k})|^2\,\langle 0|0\rangle$$

$$'' \quad \left[\hat{a}_{\vec{k}',0}\,,\ \hat{a}^{+}_{\vec{k},0}\right] = -(2\pi)^3\,2\varepsilon_k\,\delta(\vec{k}-\vec{k}')$$

$\Rightarrow$ negative norm states!

$$\boxed{H = \int\frac{d^3k}{(2\pi)^3 2\varepsilon_k}\,\varepsilon_k\left[\sum_{\lambda=1}^{3}\hat{a}^{+}_{\vec{k},\lambda}\,\hat{a}_{\vec{k},\lambda} - \hat{a}^{+}_{\vec{k},0}\,\hat{a}_{\vec{k},0}\right]} \sim \text{negative energy problem} \ldots$$

$\Rightarrow$ way out: demand that while $\partial_\mu A^\mu \neq 0$ at the operator level, for physical states one has

$$\partial_\mu A^{(+)\mu} | \psi \rangle = 0$$

where (+) denotes positive-energy part of A_μ (the term with $\hat{a}_{\vec{k},\lambda}$).

$$\langle \psi | \partial_\mu A^\mu | \psi \rangle = \langle \psi | \partial_\mu A^{(+)\mu} + \partial_\mu A^{(-)\mu} | \psi \rangle = 0.$$

(Gupta & Bleuler method) $\langle \psi | \partial_\mu A^{(-)\mu} = [\partial_\mu A^{(+)\mu} |\psi\rangle]^+ \Rightarrow$

$$\Rightarrow \partial_\mu A^{(+)\mu} | \psi \rangle = 0 \Rightarrow \sum_{\lambda=0}^{3} k^\mu \varepsilon_\mu^\lambda(\vec{k}) \, \hat{a}_{\vec{k},\lambda} | \psi \rangle = 0$$

$$\Rightarrow \left(k^\mu \varepsilon_\mu^0 \, \hat{a}_{\vec{k},0} + k^\mu \varepsilon_\mu^3 \, \hat{a}_{\vec{k},3} \right) | \psi \rangle = 0$$

as $\varepsilon^{1,2} \cdot k = 0$. Now $k \cdot \varepsilon^0 = k^0 = -k \cdot \varepsilon^3$

$$\Rightarrow \left(\hat{a}_{\vec{k},0} - \hat{a}_{\vec{k},3} \right) | \psi \rangle = 0$$

$\Rightarrow$ physical states are mixtures of longitudinal & time-like photons

$$\Rightarrow \langle \psi | \hat{a}^+_{\vec{k},0} \, \hat{a}_{\vec{k},0} | \psi \rangle = \langle \psi | \hat{a}^+_{\vec{k},3} \, \hat{a}_{\vec{k},3} | \psi \rangle$$

$\Rightarrow$ only transverse photons contribute in H:

$$\langle \psi | H | \psi \rangle = \int \frac{d^3 k}{(2\pi)^3 \, 2 \varepsilon_k} \varepsilon_k \cdot \langle \psi | \sum_{\lambda=1}^{2} \hat{a}^+_{\vec{k},\lambda} \, \hat{a}_{\vec{k},\lambda} | \psi \rangle \geq 0$$

$\Rightarrow$ no negative energy problem any more.

Massive Vector Field

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu$$, m is the mass

$\to$ $\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$ Proca eqn's.

$\partial_\mu A^\mu = 0$ always $\Rightarrow$

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \sum_{\lambda=1}^{3} \varepsilon_\mu^{(\lambda)}(\vec{k}) \left[\hat{a}_{\vec{k},\lambda} e^{-ik\cdot x} + \hat{a}^+_{\vec{k},\lambda} e^{ik\cdot x} \right]$$

with $\left[\hat{a}_{\vec{k},\lambda}, \hat{a}^+_{\vec{k}',\lambda'} \right] = \delta_{\lambda\lambda'} (2\pi)^3 2\varepsilon_k \delta(\vec{k}-\vec{k}')$

$k\cdot\varepsilon^{(\lambda)} = 0$ for $\lambda = 1, 2, 3$ $\Rightarrow$ in rest frame have $k^\mu = (m, 0, 0, 0)$ $\Rightarrow$

$\varepsilon_\mu^{(1)} = (0, 1, 0, 0)$, $\varepsilon_\mu^{(2)} = (0, 0, 1, 0)$, $\varepsilon_\mu^{(3)} = (0, 0, 0, 1)$.

3 degrees of freedom (all physical)

$$H = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \varepsilon_k \sum_{\lambda=1}^{3} \hat{a}^+_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda}$$

Time evolution: $-i\partial_t A_\mu = [H, A_\mu]$ for both massive & massless vector fields.