Correlators in Free Field Theory

Scalar Field

~ correlators are one of the most important objects in field theory

~ Consider scalar field first:

Def. $\boxed{D(x-y) \equiv \langle 0 | \varphi(x) \varphi(y) | 0 \rangle}$

2-point correlation (Green) function (x & y)

Let's calculate $D(x-y)$ in a free scalar field th'y:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2} \varphi^2$$

Remember

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \left[\hat{a}_{\vec{k}} e^{-ik\cdot x} + \hat{a}^+_{\vec{k}} e^{ik\cdot x} \right]$$

$$D(x-y) = \langle 0 | \varphi(x) \varphi(y) | 0 \rangle = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \frac{d^3k'}{(2\pi)^3 2\varepsilon_{k'}} \cdot$$

$$\langle 0 | \left[\hat{a}_{\vec{k}} e^{-ik\cdot x} + \cancel{\hat{a}^+_{\vec{k}}} e^{ik\cdot x} \right] \left[\cancel{\hat{a}_{\vec{k}'}} e^{-ik'\cdot y} + \hat{a}^+_{\vec{k}'} e^{ik'\cdot y} \right] | 0 \rangle$$

$$= \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \frac{d^3k'}{(2\pi)^3 2\varepsilon_{k'}} e^{-ik\cdot x + ik'\cdot y} \langle 0 | \hat{a}_{\vec{k}} \hat{a}^+_{\vec{k}'} | 0 \rangle$$

$$\left[\hat{a}_{\vec{k}}, \hat{a}^+_{\vec{k}'} \right] = (2\pi)^3 2\varepsilon_k \delta(\vec{k}-\vec{k}')$$

$$= \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} e^{-ik\cdot(x-y)}$$

$$\boxed{\begin{aligned} D(x-y) &= \int \frac{d^3k}{(2\pi)^3 2E_k}\, e^{-ik\cdot(x-y)} \\ &= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, (2\pi)\,\delta^{(+)}(k^2-m^2) \end{aligned}}$$

where $\delta^{(+)}(k^2-m^2) = \frac{1}{2E_k}\,\delta\left(k^0-\sqrt{\vec{k}^2+m^2}\right)$; $\delta^{(+)}(k^2-m^2) = \Theta(k^0)\,\delta(k^2-m^2)$.

$(+)$ means only positive root for k^0 counts.

(Note that $\frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon} = 2\pi i\,\delta(x) \Rightarrow$

$$2\pi\,\delta(x) = \frac{i}{x+i\varepsilon} - \frac{i}{x-i\varepsilon} = i\,2\,\mathrm{Re}\left(\frac{1}{x+i\varepsilon}\right).)$$

Def. Time-ordered product:

$$\boxed{T\,\phi(x)\,\phi(y) \equiv \Theta(x^0-y^0)\,\phi(x)\,\phi(y) + \Theta(y^0-x^0)\,\phi(y)\,\phi(x)}.$$

(the "earlier" operator is always on the right).

Def. Feynman propagator:

$$\boxed{D_F(x-y) \equiv \langle 0|\, T\,\phi(x)\,\phi(y)\,|0\rangle}$$

$$D_F(x-y) = \Theta(x^0-y^0)\,D(x-y) + \Theta(y^0-x^0)\,D(y-x).$$