Last time: finished quantizing real vector field in Lorenz gauge: $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{2} (\partial_\mu A^\mu)^2$ $\boxed{\lambda = 1}$ Feynman gauge

$$[A_\mu(\vec{x},t), \pi_\nu(\vec{y},t)] = i g_{\mu\nu} \delta(\vec{x}-\vec{y})$$

all others zero

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \sum_{\lambda=0}^{3} \varepsilon_\mu^{(\lambda)}(\vec{k}) \left[\hat{a}_{\vec{k},\lambda} e^{-ik\cdot x} + \hat{a}^+_{\vec{k},\lambda} e^{ik\cdot x} \right]$$

$$[\hat{a}_{\vec{k},\lambda}, \hat{a}^+_{\vec{k}',\lambda'}] = -g_{\lambda\lambda'} (2\pi)^3 2\varepsilon_k \delta(\vec{k}-\vec{k}')$$

Problem: $\langle 1, \lambda=0 | 1, \lambda=0 \rangle < 0$ ~ negative norm state

$$H = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \varepsilon_k \left[\sum_{\lambda=1}^{3} \hat{a}^+_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda} - \hat{a}^+_{\vec{k},0} \hat{a}_{\vec{k},0} \right]$$ ~ negative energy

Resolution: require that for physical states $|\psi\rangle$:

$$\partial_\mu A^{(+)\mu} |\psi\rangle = 0 \Rightarrow \left(\hat{a}_{\vec{k},0} - \hat{a}_{\vec{k},3} \right) |\psi\rangle = 0$$

$$\Rightarrow \langle \psi | H | \psi \rangle = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \varepsilon_k \langle \psi | \sum_{\lambda=1}^{2} \hat{a}^+_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda} | \psi \rangle \geq 0.$$

$\Rightarrow$ positive energy!

$|1, \lambda=0\rangle \propto \hat{a}^+_{\vec{k},0} |0\rangle$ is not physical as

$\left(\hat{a}_{\vec{k},0} - \hat{a}_{\vec{k},3} \right) |1, \lambda\rangle \neq 0.$ $\Rightarrow$ no negative norm problem

Massive Vector Field

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu$$

$\Rightarrow$ the same story for quantization

$\Rightarrow$ 3 physical polarizations (unlike 2 pol's for massless fields).

Correlators in Free Field Theory (cont'd)

Scalar Field (cont'd)

Def. $\boxed{D(x-y) \equiv \langle 0 | \varphi(x)\, \varphi(y) | 0 \rangle}$

Plugged in $\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} \left[\hat{a}_{\vec{k}}\, e^{-ik\cdot x} + \hat{a}^+_{\vec{k}}\, e^{ik\cdot x} \right]$

to get

$$D(x-y) = \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k}\, e^{-ik\cdot(x-y)} = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)} (2\pi)\, \delta^{(+)}(k^2 - m^2).$$

$$\boxed{\begin{aligned} D(x-y) &= \int \frac{d^3k}{(2\pi)^3 2\mathcal{E}_k} e^{-ik\cdot(x-y)} \\ &= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} (2\pi)\, \delta^{(+)}(k^2-m^2) \end{aligned}}$$

where $\delta^{(+)}(k^2-m^2) = \frac{1}{2\mathcal{E}_k}\, \delta\left(k^0 - \sqrt{\vec{k}^2+m^2}\right)$; $\delta^{(+)}(k^2-m^2) = \Theta(k^0)\,\delta(k^2-m^2)$.

(+) means only positive root for k^0 counts.

(Note that $\frac{1}{x-i\varepsilon} - \frac{1}{x+i\varepsilon} = 2\pi i\,\delta(x) \Rightarrow$

$$2\pi\,\delta(x) = \frac{i}{x+i\varepsilon} - \frac{i}{x-i\varepsilon} = i\; 2\,\mathrm{Re}\left(\frac{1}{x+i\varepsilon}\right).\Big)$$

Def. Time-ordered product:

$$\boxed{T\,\phi(x)\,\phi(y) \equiv \Theta(x^0-y^0)\,\phi(x)\,\phi(y) + \Theta(y^0-x^0)\,\phi(y)\,\phi(x)}.$$

(the "earlier" operator is always on the right).

Def. Feynman propagator:

$$\boxed{D_F(x-y) \equiv \langle 0|\, T\,\phi(x)\,\phi(y)\,|0\rangle}$$

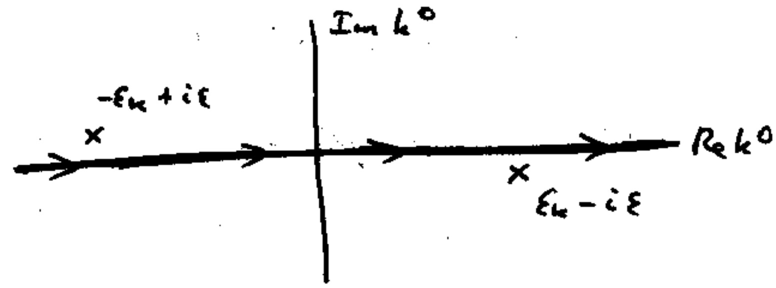
$$D_F(x-y) = \Theta(x^0-y^0)\, D(x-y) + \Theta(y^0-x^0)\, D(y-x).$$

Claim: $$\boxed{D_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)} \frac{i}{k^2-m^2+i\varepsilon}}$$

Check: $$D_F(x-y) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{i e^{-ik^0(x^0-y^0)}}{(k^0)^2 \underbrace{-\vec{k}^2-m^2}_{-\varepsilon_k^2}+i\varepsilon}$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{i e^{-ik^0(x^0-y^0)}}{(k^0-\varepsilon_k+i\varepsilon)(k^0+\varepsilon_k-i\varepsilon)}$$

The contour is shown here:



$$\frac{i}{k^2-m^2+i\varepsilon} \leftarrow \text{Feynman prescription}$$

close contour below

$$D_F(x-y) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \cdot \Big\{ \theta(x^0-y^0)\, (-\cancel{2\pi} i)\, \frac{i}{2\varepsilon_k} \cdot e^{-i\varepsilon_k(x^0-y^0)} + \theta(y^0-x^0)\, \cancel{2\pi} i\, \frac{i}{-2\varepsilon_k}\, e^{i\varepsilon_k(x^0-y^0)} \Big\} \frac{1}{\cancel{2\pi}} =$$

close contour above

$$= \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k} \Big\{ \theta(x^0-y^0)\, e^{-ik\cdot(x-y)} + \theta(y^0-x^0)\, e^{ik\cdot(x-y)} \Big\}$$

(do $\vec{k} \to -\vec{k}$ in this term)

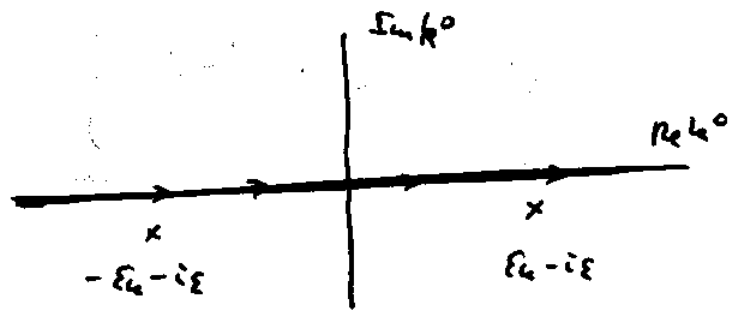
$$= \theta(x^0-y^0)\, D(x-y) + \theta(y^0-x^0)\, D(y-x) = D_F(x-y)$$

as claimed!

Other useful objects:

$$\int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik\cdot(x-y)}}{k^2-m^2+i\varepsilon k^0} = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \cdot$$

$$\cdot \frac{i\, e^{-ik^0(x^0-y^0)}}{(k^0-E_k+i\varepsilon)(k^0+E_k+i\varepsilon)}$$



Now both poles are in lower half-plane => can only close the contour below.

$$= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \frac{1}{\cancel{2\pi}} \cdot \cancel{(-2\pi i)\cdot i}\cdot \theta(x^0-y^0) \left\{ \frac{1}{2E_k} e^{-iE_k(x^0-y^0)} \right.$$

$$\left. - \frac{1}{2E_k} e^{iE_k(x^0-y^0)} \right\} = \theta(x^0-y^0)\left[D(x-y) - D(y-x)\right]$$

$$= \theta(x^0-y^0)\left[\langle 0|\varphi(x)\varphi(y)|0\rangle - \langle 0|\varphi(y)\varphi(x)|0\rangle\right]$$

$$= \theta(x^0-y^0)\,\langle 0|\left[\varphi(x),\varphi(y)\right]|0\rangle$$

Def. Retarded Green function (cf. E&M):

$$\boxed{D_R(x-y) \equiv \theta(x^0-y^0)\langle 0|[\varphi(x),\varphi(y)]|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2-m^2+i\varepsilon k^0}}$$

Retarded = causal, $\neq 0$ only in the future light cone.

Let's calculate it for massless ($m=0$) particles:

$$\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2+i\varepsilon k^0} = (\text{same as above}) =$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \theta(x^0-y^0) \left\{ \frac{1}{2\varepsilon_k} e^{-i(\varepsilon_k - i\varepsilon)(x^0-y^0)} \right.$$

$$\left. - \frac{1}{2\varepsilon_k} e^{+i(\varepsilon_k+i\varepsilon)(x^0-y^0)} \right\} \quad \text{with } \varepsilon_k = |\vec{k}| \text{ (note } i\varepsilon\text{'s).}$$

$$D_R(x-y) = \theta(x^0-y^0) \int_0^\infty k^2 dk \cdot \int_{-1}^{1} d\cos\theta \cdot \frac{1}{(2\pi)^2} \frac{1}{2k} \cdot e^{ik|\vec{x}-\vec{y}|\cos\theta} \cdot$$

$$\cdot \theta(x^0-y^0) \left[e^{-i\underset{k}{\varepsilon_k}(x^0-y^0-i\varepsilon)} - e^{i\underset{k}{\varepsilon_k}(x^0-y^0+i\varepsilon)} \right] =$$

$$= \frac{1}{8\pi^2} \theta(x^0-y^0) \int_0^\infty dk \cdot \cancel{k} \cdot \frac{1}{i\cancel{k}|\vec{x}-\vec{y}|} \left[e^{ik|\vec{x}-\vec{y}|} - e^{-ik|\vec{x}-\vec{y}|} \right]$$

$$\cdot \left[e^{-ik(x^0-y^0-i\varepsilon)} - e^{ik(x^0-y^0+i\varepsilon)} \right] = \frac{1}{8\pi^2 i} \frac{1}{|\vec{x}-\vec{y}|} \theta(x^0-y^0)$$

$$\cdot \int_0^\infty dk \left[e^{ik|\vec{x}-\vec{y}|} - e^{-ik|\vec{x}-\vec{y}|} \right] \left[e^{-ik(x^0-y^0)} - e^{ik(x^0-y^0)} \right] \cdot$$

$$\cdot e^{-k\varepsilon} = \frac{1}{8\pi^2 i} \frac{\theta(x^0-y^0)}{|\vec{x}-\vec{y}|} \left\{ \frac{-1}{i(|\vec{x}-\vec{y}|-(x^0-y^0)+i\varepsilon)} + \right.$$

$$\left. + \frac{1}{i(|\vec{x}-\vec{y}|+(x^0-y^0)+i\varepsilon)} + \frac{1}{i(-|\vec{x}-\vec{y}|-(x^0-y^0)+i\varepsilon)} - \frac{1}{i(-|\vec{x}-\vec{y}|+(x^0-y^0)+i\varepsilon)} \right\}$$

$$=\left(\text{using } \frac{1}{x+i\varepsilon}-\frac{1}{x-i\varepsilon}=-2\pi i\,\delta(x)\right)=$$

$$=\frac{1}{8\pi^2 i}\,\theta(x^0-y^0)\,\frac{1}{|\vec{x}-\vec{y}|}\left\{\frac{1}{\cancel{i}}\,2\pi\cancel{i}\,\delta\left(|\vec{x}-\vec{y}|-(x^0-y^0)\right)\right.$$

$$\left.+\frac{1}{i}(-2\pi i)\underbrace{\delta\left(|\vec{x}-\vec{y}|+(x^0-y^0)\right)}_{\overset{\shortparallel}{0}\text{ as } x^0>y^0}\right\}=\frac{-i}{4\pi}\,\theta(x^0-y^0)\,.$$

$$\cdot\frac{1}{|\vec{x}-\vec{y}|}\,\delta\left(|\vec{x}-\vec{y}|-(x^0-y^0)\right)=\frac{-i}{2\pi}\,\theta(x^0-y^0)\,\delta\left((x-y)^2\right).$$

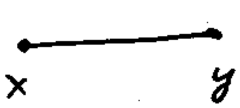
$$\Rightarrow\ D_R(x-y)\Big|_{m=0}=\frac{-i}{2\pi}\,\theta(x^0-y^0)\,\delta\left((x-y)^2\right)$$

All propagators are Green functions: e.g.

$$(\Box+m^2)\cdot D_{\substack{R\\(F)}}(x-y)=(\Box+m^2)\int\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x-y)}\,\frac{i}{k^2-m^2+i\varepsilon k^0}\ \ (+i\varepsilon)$$

$= -i\,\delta(x-y)$ ~ Green function of Klein-Gordon operator

$$\boxed{(\Box+m^2)\,D_{R,F}(x-y)=-i\,\delta(x-y)}$$



~ propagator describes propagation of particle from x to y (or y to x ~ same)

Dirac Field

Feynman propagator is

$$S_F(x-y) = \langle 0 | T\, \psi(x)\, \bar{\psi}(y) | 0 \rangle$$

where time-ordering is

$$T\, \psi_\alpha(x)\, \bar{\psi}_\beta(y) = \theta(x^0-y^0)\, \psi_\alpha(x)\, \bar{\psi}_\beta(y) - \theta(y^0-x^0)\, \bar{\psi}_\beta(y)\, \psi_\alpha(x).$$

Plug in

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \sum_{r=1}^{2} \left\{ \hat{b}_{\vec{k},r}\, u_r(\vec{k})\, e^{-ik\cdot x} + \hat{d}^{\dagger}_{\vec{k},r}\, v_r(\vec{k})\, e^{ik\cdot x} \right\}$$

into $\langle 0 | \psi(x)_\alpha\, \bar{\psi}(y)_\beta | 0 \rangle \overset{\text{only } \hat{b}\hat{b}^\dagger \text{ contribute}}{=} \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \sum_{r=1}^{2} u_{r,\alpha}(\vec{k}) \cdot$

spinor indices

$\cdot\, \bar{u}_{r,\beta}(\vec{k})\, e^{-ik\cdot(x-y)} = \left(\text{as } \sum_r u_{r,\alpha}(\vec{k})\, \bar{u}_{r,\beta}(\vec{k}) = (\gamma\cdot k + m)_{\alpha\beta} \right) =$

$$= \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} (\gamma\cdot k + m)_{\alpha\beta}\, e^{-ik\cdot(x-y)}$$

Similarly $\langle 0 | \bar{\psi}(y)_\beta\, \psi_\alpha(x) | 0 \rangle \overset{\hat{d}\hat{d}^\dagger \text{ contribute}}{=} \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} \sum_{r=1}^{2} \bar{v}_{r,\beta}(\vec{k}) \cdot$

$\cdot\, v_{r,\alpha}(\vec{k})\, e^{ik\cdot(x-y)} = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k} (\gamma\cdot k - m)_{\alpha\beta}\, e^{ik\cdot(x-y)}$

$$S_F(x-y) = \theta(x^0-y^0) \langle 0 | \psi(x)\, \bar{\psi}(y) | 0 \rangle - \theta(y^0-x^0) \langle 0 | \bar{\psi}(y)\, \psi(x) | 0 \rangle$$