<u>Last time</u>: finished talking about Feynman propagators in free field theories:

<u>Dirac field</u>:
$$S_F(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i(\gamma\cdot k + m)}{k^2 - m^2 + i\varepsilon}$$

(two contributions)

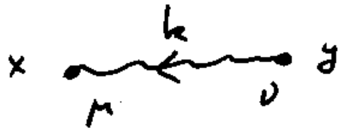
<u>Gauge field</u>: in $\partial_\mu A^\mu = 0$ gauge:

$$D_{\mu\nu}(x-y) \equiv \langle 0 | T A_\mu(x) A_\nu(y) | 0 \rangle =$$
$$= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{-i}{k^2+i\varepsilon}\left[g_{\mu\nu} + (\lambda-1)\frac{k_\mu k_\nu}{k^2}\right]$$

Feynman gauge $\lambda = 1$ $\Rightarrow$ $\tilde{D}_{\mu\nu}(k) = \frac{-i}{k^2+i\varepsilon} g_{\mu\nu}$

Landau gauge $\lambda = 0$ $\Rightarrow$ $\tilde{D}_{\mu\nu}(k) = \frac{-i}{k^2+i\varepsilon}\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right].$

where $D_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \tilde{D}_{\mu\nu}(k).$



Interacting Fields and Feynman Diagrams (cont'd)

- interesting field theories have interactions
- examples: φ^3 theory, QED, φ^4 theory:

$$\mathcal{L}_{\varphi^4} = \underbrace{\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2}_{\text{free theory}} \underbrace{- \frac{\lambda}{4!}\varphi^4}_{\text{interaction term.}}$$

Interacting Fields and Feynman Diagrams

The interesting and physically relevant field theories have interaction terms.

Examples:

$$\mathcal{L}_{QED} = \underbrace{\bar{\psi}\left[i\not{\partial} - m\right]\psi}_{\text{free Dirac field}} - \underbrace{\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{\text{free gauge field}} - \underbrace{\overset{\text{coupling constant}}{\overset{\downarrow}{e}}\, \bar{\psi} \gamma^\mu \psi A_\mu}_{\text{interaction term}}$$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2} \varphi^2 - \underbrace{\overset{\text{coupling}}{\frac{\lambda}{4!}} \varphi^4}_{\text{interaction}}$$

"phi to the fourth" theory

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2} \varphi^2 - \underbrace{\frac{\lambda}{3!} \varphi^3}_{\text{interactions}}$$

"phi - cubed" theory

Take φ^4 theory:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2} \varphi^2 - \frac{\lambda}{4!} \varphi^4$$ & can do canonical quantization: $[\varphi(\vec{x},t), \pi(\vec{y},t)] = i\delta(\vec{x}-\vec{y})$

$\Rightarrow$ EOM are

$$(\Box + m^2)\varphi = -\frac{\lambda}{3!}\varphi^3$$ $\Rightarrow$ can not decompose the field φ into creation & annihilation operators $(\hat{a}^\dagger, a)$...

$\Rightarrow$ can not calculate anything...

Interaction Picture & Correlation Functions

Want to calculate $\langle \Psi_0 | T\, \phi(x)\, \phi(y) | \Psi_0 \rangle$

in φ^4 theory. Do not have φ's in terms of $\hat{a}, \hat{a}^+$

$\Rightarrow$ how do we calculate this?

$|\Psi_0\rangle$ ~ ground state of interaction theory (vacuum).

In general $|\Psi_0\rangle \neq |0\rangle$ ~ not the same as in free theory.

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi\, \partial^\mu \varphi - \frac{m^2}{2} \varphi^2 - \frac{\lambda}{4!} \varphi^4$$

$$\Rightarrow H = \int d^3x \Big[\underbrace{\frac{1}{2} \pi^2 + \frac{1}{2} (\vec{\nabla} \varphi)^2 + \frac{m^2}{2} \varphi^2}_{\text{free Ham} \equiv H_0} + \underbrace{\frac{\lambda}{4!} \varphi^4}_{\substack{\text{interaction Ham} \\ = H_{int}}} \Big]$$

$\Rightarrow$ $H = H_0 + H_{int}$ (Heisenberg or Schrodinger picture)

<u>Interaction picture:</u> $-i\hbar \frac{d}{dt} \langle \psi | \hat{O} | \psi \rangle = \langle \psi | [\hat{H}, \hat{O}] | \psi \rangle$, $\hbar = 1$

$\Rightarrow -i \frac{d\hat{O}}{dt} = [\hat{H}_0, \hat{O}] \Rightarrow \hat{O}_I(\vec{x}, t) = e^{i\hat{H}_0 t}\, \hat{O}_S(\vec{x})\, e^{-i\hat{H}_0 t}$

$i \frac{d}{dt} |\psi\rangle_I = \hat{H}_I |\psi\rangle_I \Rightarrow \hat{H}_I$ is the interaction Ham. in the interaction picture, $\hat{H}_I = e^{i\hat{H}_0 t}\, \hat{H}_{int}\, e^{-i\hat{H}_0 t}$.

($H = H_0 + H_I = e^{iH_0 t} (H_0 + H_{int}) e^{-iH_0 t}$, just like other operators in interac. pict.)

$\Rightarrow$ in the interaction picture the field evolves with H_0 $\Rightarrow$ $-i\,\partial_0 \varphi_I = [H_0, \varphi_I]$ $\Rightarrow$

$$\varphi_I(\vec{x},t) = e^{iH_0 t}\,\varphi_S(\vec{x})\,e^{-iH_0 t}$$

Also, combining $-i\,\partial_0 \varphi_I = [H_0, \varphi_I]$ with canonical equal-time commutation relations $[\varphi_I(\vec{x},t), \pi_I(\vec{x},t)] = i\delta(\vec{x}-\vec{y})$

$\Rightarrow$ φ_I satisfies free(!) Klein-Gordon equation

$$(\Box + m^2)\,\varphi_I = 0$$

$$\Rightarrow \varphi_I(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3\, 2\varepsilon_k}\left[\hat{a}_{\vec{k}}\, e^{-ik\cdot x} + \hat{a}^{\dagger}_{\vec{k}}\, e^{ik\cdot x}\right]$$

"as usual": $\varphi_I(\vec{x},t) \sim$ just like a free field.

Def. Unitary operator

$$\boxed{U(t,t') = e^{iH_0(t-t')}\, e^{-iH(t-t')}}$$

(time evolution operator)

$$\varphi_H(t,\vec{x}) = e^{iHt}\,\varphi_S(\vec{x})\,e^{-iHt} = e^{iHt}\,e^{-iH_0 t}\,\varphi_I(\vec{x},t)$$

$$\cdot\, e^{iH_0 t}\,e^{-iHt} = U^{\dagger}(t,t_0)\,\varphi_I(\vec{x},t)\,U(t,t_0) \quad (t_0 \sim \text{arbitrary reference time})$$

$$\Rightarrow \varphi_H(\vec{x},t) = U^{\dagger}(t,t_0)\,\varphi_I(\vec{x},t)\,U(t,t_0)$$

Note that $i \frac{\partial}{\partial t} U(t,t') = i\Big[e^{iH_0(t-t')} iH_0 e^{-iH(t-t')}$

$- e^{iH_0(t-t')} (-iH) e^{-iH(t-t')} \Big] = e^{iH_0(t-t')} (H - H_0) e^{-iH(t-t')}$

$= e^{iH_0(t-t')} H_{int} e^{-iH(t-t')} = \underbrace{e^{iH_0(t-t')} H_{int} e^{iH_0(t-t')}}_{H_I \sim \text{interaction Ham in the int. picture.}}$

$\cdot \underbrace{e^{-iH_0(t-t')} e^{-iH(t-t')}}_{U(t,t')}$

(Aside: H in general depends on $t \Rightarrow H = H(t) \sim$ Heisenberg picture? $\Rightarrow -i\partial_t \hat{H} = [\hat{H}, \hat{H}] \Rightarrow \hat{H}$ is time-independent in Heisenberg picture. In Schrodinger picture $\partial_t \hat{H} = 0$ too, as all operators are time-independent.)

φ^4: $H_I = \int d^3x \frac{\lambda}{4!} \varphi_I^4$

$$\Rightarrow \boxed{i \frac{\partial}{\partial t} U(t,t') = H_I(t) \quad U(t,t')}$$

Note that $i \frac{d}{dt} |\psi\rangle_I = \hat{H}_I |\psi\rangle_I \quad \Rightarrow$

$$|\psi(t)\rangle_I = U(t,t') \, |\psi(t')\rangle_I$$

$\Rightarrow U(t,t')$ provides time evolution of state $|\psi(t)\rangle_I$

$\Rightarrow$ time evolution operator!

Let us now find $U(t,t')$ in terms of H_I:

solve $i\partial_t U(t,t') = H_I(t) \, U(t,t')$