Last time:

Interaction Picture & Correlation Functions

(cont'd)

want $\langle \Psi_0 | T\, \varphi(x)\, \varphi(y) | \Psi_0 \rangle$

$$\mathcal{L} = \underbrace{\frac{1}{2}\partial_\mu \varphi\, \partial^\mu \varphi - \frac{m^2}{2}\varphi^2}_{\text{free}} - \underbrace{\frac{\lambda}{4!}\varphi^4}_{\text{int.}} \quad \sim \varphi^4 \text{ theory}$$

$$H = H_0 + H_{int}$$

$$-i\frac{d}{dt}\langle \psi | \hat{O} | \varphi \rangle = \langle \psi | [\hat{H}, \hat{O}] | \varphi \rangle$$

$\Rightarrow$ can solve by $\hat{H} = \hat{H}_1 + \hat{H}_2$

$$-i\frac{d\hat{O}}{dt} = [\hat{H}_1, \hat{O}], \quad i\frac{d}{dt}|\psi\rangle = \hat{H}_2 |\psi\rangle .$$

$\hat{H}_1 = H$, $\hat{H}_2 = 0$ ~ Heisenberg picture.

(note: $\hat{H}_H = e^{i\hat{H}t}\, \hat{H}\, e^{-i\hat{H}t} = \hat{H}$)

$\hat{H}_1 = 0$, $\hat{H}_2 = H$ ~ Schrödinger picture

(note: $\hat{H}_S = \hat{H} \Rightarrow \hat{H}_S = \hat{H}_H$.)

$\hat{H}_1 = \hat{H}_0$, $\hat{H}_2 = \hat{H}_I$ ~ Interaction picture:

note $\hat{H}_0$ is always t-independent $\Rightarrow$

$$\hat{H}_{\text{interaction}} = e^{iH_0 t}\left(\hat{H}_0 + \hat{H}_{int}\right)e^{-iH_0 t} = \hat{H}_0 + \underbrace{e^{iH_0 t} H_{int} e^{-iH_0 t}}_{H_I}$$

$$H_I = e^{iH_0 t} H_{int} e^{-iH_0 t}$$

$$\varphi_I = e^{iH_0 t} \varphi_s e^{-iH_0 t}$$ ~ evolves in time like a free field

$$(\Box + m^2)\varphi_I = 0 \Rightarrow$$

can decompose into $\hat{a}, \hat{a}^+$:

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\varepsilon_k}\left[\hat{a}_{\vec{k}} e^{-ik\cdot x} + \hat{a}^+_{\vec{k}} e^{ik\cdot x}\right]$$

Def. $$\boxed{U(t,t') \equiv e^{iH_0(t-t')} e^{-iH(t-t')}}$$

$$\varphi_H(\vec{x},t) = U^+(t,t_0)\, \varphi_I(\vec{x},t)\, U(t,t_0)$$

$$i\frac{\partial}{\partial t} U(t,t') = H_I(t)\, U(t,t')$$ ~ proved this

$\Rightarrow$ as $i\frac{d}{dt}|\psi\rangle_I = \hat{H}_I(t)\,|\psi_I\rangle \Rightarrow$

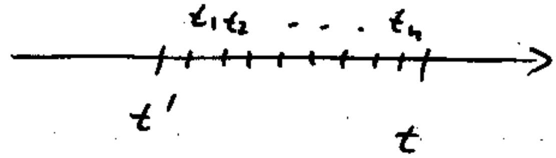
$$\Rightarrow |\psi(t)\rangle_I = U(t,t')\,|\psi(t')\rangle_I$$

$\Rightarrow U(t,t')$ is the time <u>evolution</u> operator for states!

$$\boxed{\partial_t U(t,t') = -i H_I(t)\, U(t,t'), \quad U(t,t) = \mathbb{1}}$$

if $H_I(t) = H_I$ (time-indep.) $\Rightarrow U = e^{-i H_I t}$.

Not so simple in general. Split time interval in short steps Δt:



In each step get:

$$U(t+\Delta t, t) = \left(1 - i\, H_I(t)\, \Delta t\right) \overbrace{U(t,t)}^{=\mathbb{1}}.$$

$$U(t,t') = \left(1 - i\Delta t \cdot H_I(t_n)\right)\left(1 - i\Delta t\, H_I(t_{n-1})\right) \cdots$$

$$\cdots \left(1 - i\Delta t\, H_I(t_1)\right)\left(1 - i\Delta t\, H_I(t)\right) \underbrace{U(t,t)}_{\mathbb{1}} \rightarrow$$

$$\rightarrow \lim_{n\to\infty} \prod_{i=1}^{n} \left(1 - i\Delta t\, H_I(t_i)\right) = \lim_{n\to\infty} \prod_{i=1}^{n} \Big[1 -$$

$$- i\, \frac{t-t'}{n}\, H_I(t_i) \Big]$$

$\Rightarrow$ would like to simply exponentiate, but $H_I(t_i)$ do not commute

Def. time-ordered exponential:

$$T \exp\left\{-i \int_{t'}^{t} dt''\, H_I(t'')\right\} \equiv$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{1}{n!} (-i)^n \int_{t'}^{t} dt_1 \ldots dt_n \; T\{ H_I(t_1) \ldots H_I(t_n) \}.}$$

For instance, for $n=2$ have

$$\frac{1}{2!} \int_{t'}^{t} dt_1\, dt_2 \; T\{H_I(t_1)\, H_I(t_2)\} = \frac{1}{2!} \int_{t'}^{t} dt_1\, dt_2 \cdot [\theta(t_1 - t_2) \cdot$$

$$\cdot H_I(t_1)\, H_I(t_2) + \theta(t_2 - t_1)\, H_I(t_2)\, H_I(t_1)] = \frac{1}{2!} \left[\int_{t'}^{t} dt_1 \cdot \int_{t'}^{t_1} dt_2 \cdot \right.$$

$$\left. \cdot H_I(t_1)\, H_I(t_2) + \underbrace{\int_{t'}^{t} dt_1 \int_{t_1}^{t} dt_2}_{\int_{t'}^{t} dt_2 \int_{t'}^{t_2} dt_1} \; H_I(t_2)\, H_I(t_1) \right]$$

$\Rightarrow$ swap $t_1 \leftrightarrow t_2 \Rightarrow$ doubles the first term

$$= \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \; H_I(t_1)\, H_I(t_2)$$

later times are to the left.

$$\Rightarrow \boxed{T\exp\left\{ -i \int_{t'}^{t} dt''\, H_I(t'') \right\} = \sum_{n=0}^{\infty} (-i)^n \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \ldots \int_{t'}^{t_{n-1}} dt_n \cdot H_I(t_1) \ldots H_I(t_n)}$$

$$\Rightarrow \partial_t \; T\exp\left\{ -i \int_{t'}^{t} dt''\, H_I(t'') \right\} = -i\, H_I(t) \cdot T\exp\left\{ -i \int_{t'}^{t} dt''\, H_I(t'') \right\}$$

$$\Rightarrow \boxed{U(t,t') = T\exp\left\{ -i \int_{t'}^{t} dt''\, H_I(t'') \right\}}$$

Note also that

$$\boxed{T \exp\left\{-i\int_{t'}^{t} dt'' H_I(t'')\right\} = \lim_{n\to\infty} \prod_{i=1}^{n} \left[1 - i\Delta t\, H_I(t_i)\right]}$$

$\Rightarrow$ pick up a factor of $1 - i\Delta t\, H_I(t_i)$ at each infinitesimal step, factors don't commute $\Rightarrow$

$\Rightarrow$ exponentiation is non-trivial.

Note that $\boxed{U(t_1,t_2)\, U(t_2,t_3) = U(t_1,t_3).}$

if $t_1 = t_3 \Rightarrow \quad U(t_1,t_2)\, U(t_2,t_1) = \mathbb{1}.$

Note also That the time-ordered definition of exponent yields:

$$\boxed{U(t_1,t_2)^{+} = U(t_2,t_1)}$$

$\Rightarrow \boxed{U(t_1,t_2)\, U^{+}(t_1,t_2) = \mathbb{1}} \Rightarrow$

time evolution is <u>unitary</u> (norm preserving) as expected (probability does not disappear):

$$\langle\psi(t_1)|\psi(t_1)\rangle = \left|U(t_1,t_2)|\psi(t_2)\rangle\right|^2 = \langle\psi(t_2)|\, U^{+}(t_1,t_2)\, U(t_1,t_2)|\psi(t_2)\rangle$$

$= \langle\psi(t_2)|\psi(t_2)\rangle \Rightarrow$ norm is The same at all times

We want to find ${}_H\langle \Psi_0 | T\, \varphi_H(x)\, \varphi_H(y) | \Psi_0 \rangle_H$.

Using $\varphi_H(x) = U^+(t, t_0)\, \varphi_I(x)\, U(t, t_0)$ we write

$${}_H\langle \Psi_0 | \varphi_H(x)\, \varphi_H(y) | \Psi_0 \rangle_H = {}_H\langle \Psi_0 | U^+(x^0, t_0)\, \varphi_I(x) \cdot$$

$$\underbrace{\cdot\, U(x^0, t_0)\, \underbrace{U^+(y^0, t_0)}_{U(t_0, y^0)}}_{U(x^0, y^0)}\, \varphi_I(y)\, U(y^0, t_0) | \Psi_0 \rangle_H =$$

$$= {}_H\langle \Psi_0 | U^+(x^0, t_0)\, \varphi_I(x)\, U(x^0, y^0)\, \varphi_I(y)\, U(y^0, t_0) | \Psi_0 \rangle_H$$

$$|\Psi_0(t)\rangle_I = U(t, t_0)\, |\Psi_0(t_0)\rangle_I = U(t, t_0) \cdot |\Psi_0\rangle_H$$

$$\Rightarrow |\Psi_0\rangle_H = U^+(t, t_0)\, |\Psi_0(t)\rangle_I = U(t_0, t)\, |\Psi_0(t)\rangle_I .$$

Pick $t = -\infty \Rightarrow |\Psi_0\rangle_H = U(t_0, -\infty)\, |\Psi_0(-\infty)\rangle_I$.

$$\Rightarrow {}_H\langle \Psi_0 | = {}_I\langle \Psi_0(t) |\, U(t, t_0) = {}_I\langle \Psi_0(+\infty) |\, U(+\infty, t_0).$$

Assume that initially we have perturbative vacuum (at $t = -\infty$), such that (see Peskin pp. 86-87)

can start from $\forall$ state

$$|\Psi_0(-\infty)\rangle_I = |0\rangle .$$

Then ${}_I\langle \Psi_0(+\infty) | = {}_I\langle \Psi_0(-\infty) |\, U(-\infty, +\infty) = \langle 0 |\, U(-\infty, +\infty)$

(Note that to make sense of

$$U(-\infty, +\infty) = T \exp\left\{ -i \int_{-\infty}^{\infty} dt\, H_I(t) \right\}$$

need to multiply $\pm\infty$ by $(1 - i\varepsilon)$ => makes it finite!)

We have: ${}_H\langle \Psi_0 | \varphi_H(x)\, \varphi_H(y) | \Psi_0 \rangle_H = \langle 0 | U(-\infty, +\infty)$

$\cdot\, U(+\infty, t_0)\, U(t_0, x^0)\, \varphi_I(x)\, U(x^0, y^0)\, \varphi_I(y)\, U(y^0, t_0)\cdot$

$\cdot\, U(t_0, -\infty) | 0 \rangle = \langle 0 | U(-\infty, +\infty)\, U(+\infty, x^0)\, \varphi_I(x)\cdot$

$\cdot\, U(x^0, y^0)\, \varphi_I(y)\, U(y^0, -\infty) | 0 \rangle = \sum_n \langle 0 | U(-\infty, +\infty) | n \rangle\cdot$

$\cdot \langle n | U(+\infty, x^0)\, \varphi_I(x)\, U(x^0, y^0)\, \varphi_I(y)\, U(y^0, -\infty) | 0 \rangle.$

Vacuum can only evolve into vacuum (otherwise energy / momentum are not conserved):

$$\langle 0 | U(-\infty, +\infty) | n \rangle = \delta_{n0}\, e^{i\Phi}$$

$\uparrow$ phase.

$1 = \langle 0 | 0 \rangle = \langle 0 | U(-\infty, +\infty) \cdot U(+\infty, -\infty) | 0 \rangle = \sum_n \cdot$

$\cdot \langle 0 | U(-\infty, +\infty) | n \rangle \langle n | U(+\infty, -\infty) | 0 \rangle =$

$= \left| \langle 0 | U(-\infty, +\infty) | 0 \rangle \right|^2 \Rightarrow \langle 0 | U(-\infty, +\infty) | 0 \rangle = e^{i\Phi}$

$$\Rightarrow e^{-i\Phi} = \langle 0 | U(+\infty, -\infty) | 0 \rangle = \frac{1}{\langle 0 | U(-\infty, +\infty) | 0 \rangle}$$

$$ {}_H\langle \Psi_0 | \varphi_H(x)\varphi_H(y) | \Psi_0 \rangle_H = \langle 0 | U(-\infty, +\infty) | 0 \rangle \cdot $$

$$ \cdot \langle 0 | U(+\infty, x^0)\, \varphi_I(x^0)\, U(x^0, y^0)\, \varphi_I(y^0)\, U(y^0, -\infty) | 0 \rangle $$

$$ = \frac{1}{\langle 0 | U(+\infty, -\infty) | 0 \rangle} \langle 0 | U(+\infty, x^0)\, \varphi_I(x^0)\, U(x^0, y^0)\, \varphi_I(y^0) \cdot $$

$$ \cdot U(y^0, -\infty) | 0 \rangle . $$

We therefore have

$$ \boxed{\langle \Psi_0 | \varphi(x)\varphi(y) | \Psi_0 \rangle = \frac{\langle 0 | U(+\infty, x^0)\, \varphi_I(x)\, U(x^0, y^0)\, \varphi_I(y)\, U(y^0, -\infty) | 0 \rangle}{\langle 0 | U(+\infty, -\infty) | 0 \rangle}} $$

Inserting time-ordering yields:

$$ \boxed{\langle \Psi_0 | T\, \varphi(x)\varphi(y) | \Psi_0 \rangle = \frac{\langle 0 | T \{ \varphi_I(x)\, \varphi_I(y)\, e^{-i\int_{-\infty}^{\infty} dt\, H_I(t)} \} | 0 \rangle}{\langle 0 | T\, e^{-i\int_{-\infty}^{\infty} dt\, H_I(t)} | 0 \rangle}} $$

This is also true in general:

$$ \boxed{\begin{aligned} &\langle \Psi_0 | T \{ \varphi_H(x_1)\, \varphi_H(x_2) \ldots \varphi_H(x_n) \} | \Psi_0 \rangle = \\ &= \frac{\langle 0 | T \{ \varphi_I(x_1)\, \varphi_I(x_2) \ldots \varphi_I(x_n)\, e^{-i\int_{-\infty}^{\infty} dt\, H_I(t)} \} | 0 \rangle}{\langle 0 | T\, e^{-i\int_{-\infty}^{\infty} dt\, H_I(t)} | 0 \rangle} \end{aligned}} $$

Main principle: φ_I ~ just like free field.

=> expand in H_I => get a series in powers of coupling constant => perturbation theory.

In φ^4 theory we have: $H_I(t) = \int d^3x \, \frac{\lambda}{4!} \varphi_I^4(x)$. =>

$$\int_{-\infty}^{\infty} dt \; H_I(t) = \frac{\lambda}{4!} \int d^4x \; \varphi_I^4(x).$$

Calculate two-point correlator:

$$\langle \Psi_0 | T \varphi(x) \varphi(y) | \Psi_0 \rangle = \frac{1}{\langle 0 | T \exp\left\{ -i \int d^4x \, \frac{\lambda}{4!} \varphi_I^4(x) \right\} | 0 \rangle}$$

$$\cdot \Big[\langle 0 | T \varphi_I(x) \varphi_I(y) | 0 \rangle - i \frac{\lambda}{4!} \int d^4z \, \langle 0 | T \{ \varphi_I(x) \varphi_I(y) \varphi_I^4(z) \} | 0 \rangle$$

$$- \frac{1}{2!} \left(\frac{\lambda}{4!} \right)^2 \int d^4z_1 \, d^4z_2 \, \langle 0 | T \{ \varphi_I(x) \varphi_I(y) \varphi_I^4(z_1) \varphi_I^4(z_2) \} | 0 \rangle$$

$$+ \ldots \Big]$$

=> get a series in the coupling λ

=> at lowest order get Feynman propagator

=> to evaluate the series need to know how to find terms like